# Stabilization of Incompressible Flow Problems by Riccati-Based Feedback

Eberhard Bänsch and Peter Benner

**Abstract.** We consider optimal control-based boundary feedback stabilization of flow problems for incompressible fluids. We follow an analytical approach laid out during the last years in a series of papers by Barbu, Lasiecka, Triggiani, Raymond, and others. They have shown that it is possible to stabilize perturbed flows described by Navier-Stokes equations by designing a stabilizing controller based on a corresponding linear-quadratic optimal control problem. For this purpose, algorithmic advances in solving the associated algebraic Riccati equations are needed and investigated here. The computational complexity of the new algorithms is essentailly proportional to the simulation of the forward problem.

**Mathematics Subject Classification (2000).** 76D55,93D15,93C20,15A124.

**Keywords.** flow control, feedback, Navier-Stokes equations, Riccati equation.

## 1. Introduction

The aim of this work is to develop numerical methods for the stabilization of solutions to flow problems. This is to be achieved by action of boundary control using feedback mechanisms. In recent work by Raymond [48, 49, 50] and earlier attempts by Barbu [8], Barbu and Triggiani [11] and Barbu, Lasiecka, and Triggiani [9], it is shown analytically that it is possible to construct a linear-quadratic optimal control problem associated to the linearized Navier-Stokes equation so that the resulting feedback law, applied to the instationary Navier-Stokes equation, is able to exponentially stabilize unstable solution trajectories assuming smallness of initial values.

---

The work described in this paper was supported by *Deutsche Forschungsgemeinschaft*, Priority Programme 1253, project BA1727/4-1 and BE2174/8-1 "Optimal Control-Based Feedback Stabilization in Multi-Field Flow Problems".

To be more precise, consider the following situation. The flow velocity field $v$ and pressure $\chi$ fulfill the incompressible Navier-Stokes equations

$$\partial_t v + v \cdot \nabla v - \frac{1}{Re} \Delta v + \nabla \chi = f, \tag{1a}$$

$$\operatorname{div} v = 0, \tag{1b}$$

on $Q_\infty := \Omega \times (0, \infty)$ with a bounded and connected domain $\Omega \subseteq \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma := \partial\Omega$ of class $C^4$, a Dirichlet boundary condition $v = g$ on $\Sigma_\infty := \Gamma \times (0, \infty)$, and appropriate initial conditions (the latter are discussed, e.g., in [3]).

Now assume we are given a regular solution $w$ of the stationary Navier-Stokes equations

$$w \cdot \nabla w - \frac{1}{Re} \Delta w + \nabla \chi_s = f, \tag{2a}$$

$$\operatorname{div} w = 0, \tag{2b}$$

with Dirichlet boundary condition $w = g$ on $\Gamma$. Moreover, the given flow field $w$ is assumed to be an *unstable* solution of (1).

If one can determine a Dirichlet boundary control $u$ so that the corresponding controlled system

$$\partial_t z + (z \cdot \nabla) w + (w \cdot \nabla) z + (z \cdot \nabla) z - \frac{1}{Re} \Delta z + \nabla p = 0 \quad \text{in } Q_\infty, \tag{3a}$$

$$\operatorname{div} z = 0 \quad \text{in } Q_\infty, \tag{3b}$$

$$z = bu \quad \text{in } \Sigma_\infty, \tag{3c}$$

$$z(0) = z_0 \quad \text{in } \Omega, \tag{3d}$$

is stable for initial values $z_0$ sufficiently small in an appropriate subspace $X(\Omega)$ of the space of divergence-free $L_2$ functions with $z \cdot n = 0$, called here $V_n^0(\Omega)$, then it can be shown for several situations (see below) that $z$ decreases exponentially in the norm of $X(\Omega)$ and thus the solution to the instationary Navier-Stokes equations (2) with flow field $v = w + z$, pressure $\chi = \chi_s + p$, and initial condition $v(0) = w + z_0$ in $\Omega$ is controlled to the stationary solution $w$. The initial value $z_0$ can be seen as a small perturbation of the steady-state flow $w$ and the considered problem can be interpreted as designing a feedback control in order to stabilize the perturbed flow back to the steady-state solution. Note that the operator $b$ in the boundary control formulation (3c) is the identity operator if the control acts on the whole boundary and can be considered as a restriction operator if the control is localized in part of the boundary.

The following analytical solutions to the control problem described above are discussed in the literature. For $w \in L_\infty(\Omega)$ and $z_0 \in V_n^0(\Omega) \cap L_4(\Omega) =: X(\Omega)$, the existence of a stabilizing boundary control is proved in [29], but no constructive way to derive a boundary control in feedback form is derived. In the 3D case, the existence of an exponentially stabilizing feedback control for an appropriately defined subspace $X(\Omega)$ is proved in [30].

Stabilization of the Navier-Stokes system with feedback control based on an associated linear-quadratic optimal control problem has been recently discussed by several authors. The situation as described above with a distributed control localized in an open subset $\Omega_u \subset \Omega$ of positive measure instead of a boundary control is discussed in [8, 10, 11]. The only numerical study of such an approach known to the authors is [39] and treats the special case of a rectangular driven cavity and a control with zero normal component. The problem considered above with Dirichlet boundary control, $b$ being the identity operator, and a control with zero normal component is studied in [9]. The first treatment of the problem considered above in the case when the control has nonzero normal components (which is often the relevant case in engineering applications) and is localized on parts of the boundary is given in [48, 49, 50, 3]. The stabilization results described in [48, 49, 50, 3] are constructive in that they allow the computation of a Dirichlet feedback law which stabilizes the Navier-Stokes system (1) in the sense that its solution $(v, \chi)$ is controlled to the steady-state solution $(w, \chi_s)$ associated to (2). We will briefly outline this construction for the 2D case as described in [48, 49]—the 3D case is treated similarly [50], but the derivation is quite more involved. The stabilizing feedback law is derived from a perturbed linearization of the Navier-Stokes system (3), given by

$$\partial_t z + (z \cdot \nabla)w + (w \cdot \nabla)z - \frac{1}{Re}\Delta z - \omega z + \nabla p = 0 \quad \text{in } Q_\infty, \tag{4a}$$

$$\operatorname{div} z = 0 \quad \text{in } Q_\infty, \tag{4b}$$

$$z = bu \quad \text{in } \Sigma_\infty, \tag{4c}$$

$$z(0) = z_0 \quad \text{in } \Omega, \tag{4d}$$

where $\omega > 0$ is a positive constant. The perturbation term $-\omega z$ (note the sign!) is required to achieve exponential stabilization of the feedback law. Together with the cost functional

$$J(z, u) = \frac{1}{2}\int_0^\infty \langle z, z\rangle_{L_2(\Omega)} + \langle u, u\rangle_{L_2(\Gamma)}\, dt, \tag{5}$$

the linear-quadratic optimal control problem associated to (3) becomes

$$\inf\left\{J(z,u) \mid (z,u) \text{ satisfies (4)},\ u \in L_2((0,\infty); V^0(\Gamma))\right\}, \tag{6}$$

where $V^0(\Gamma) = \{g \in L_2(\Gamma) \mid \langle g \cdot n, 1\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} = 0\}$. Then it is shown in [48, 49] that the feedback law

$$u = -R_A^{-1} b B^* \Pi P z \tag{7}$$

is exponentially stabilizing for small enough initial values $z_0$. The operators defining the feedback law are

- the linearized Navier-Stokes operator $A$;
- the Helmholtz projector $P : L_2(\Omega) \mapsto V_n^0(\Omega)$;
- the control operator $B := (\lambda_0 I - A)D_A$, where $D_A$ is the Dirichlet operator associated with $\lambda_0 I - A$ and $\lambda_0 > 0$ is a constant;

- $R_A := bD_A^*(I-P)D_Ab + I$;
- the Riccati operator $\Pi = \Pi^* \in \mathcal{L}(V_n^0(\Omega))$ which is the unique nonnegative semidefinite weak solution of the operator Riccati equation

$$0 = I + (A+\omega I)^*\Pi + \Pi(A+\omega I) - \Pi(B_\tau B_\tau^* + B_n R_A^{-1} B_n^*)\Pi, \tag{8}$$

  where $B_\tau$ and $B_n$ correspond to the projection of the control action in the tangential and normal directions derived from the control operator $B$.

Note that a simpler version of the Riccati equation and feedback control law with $R_A = I$ is discussed in [3].

Thus, the numerical solution of the stabilization problem for the Navier-Stokes equation (1) requires the numerical solution of the operator Riccati equation (8). To achieve this, we propose to proceed as in classical approaches to linear-quadratic optimal control problems for parabolic systems described, e.g., in [4, 5, 41, 42]. That is, first we discretize (4) in space using a finite element (FEM) Galerkin scheme, then solve the associated finite-dimensional algebraic Riccati equation (ARE) from which we obtain a finite-dimensional controller in feedback form as in (7). The finite-dimensional controller is then applied to (1). The stabilizing properties of such a finite-dimensional controller in case of internal control are discussed in [11] and for linear parabolic systems in [36, 45]. For the boundary control problem considered here, the approximation and convergence theory is an open problem. As far as the numerical solution of the described stabilization problems is concerned, we are only aware of the abovementioned paper [39] and preliminary results for stabilization of Kármán vortex streets presented by Raymond in [47] which, in both cases, lead to dimensions of the FEM ansatz space so that the associated ARE can still be solved with classical methods based on associated eigenvalue problems (see [14, 26, 52] for overviews of available methods).

We also would like to point out that there exists a variety of different approaches to the stabilization of flow problems based, e.g., on gradient or adjoint methods, see the recent books [1, 34] and the references therein. Another recent contribution treats the 3D Navier-Stokes system in the exterior of a bounded domain [31]. Here, a control is derived based on an optimality system involving the adjoint Oseen equations. In contrast to all these approaches, the ansatz followed here does not require the sometimes infeasible computation of gradient systems or the solution of adjoint systems. We pursue the feedback stabilization approach as it allows to incorporate current information on the perturbed flow, which is not possible when using a pre-computed open-loop control as often obtained from optimization-based methods. That is, our approach allows to also deal with perturbations "on the fly" in contrast to open-loop control schemes. Of course, in practical situations, our approach will be best utilized in order to compensate for perturbations of an optimized flow profile that is obtained through open-loop control, i.e., the feedback control loop is added to optimal control obtained by other methods. If the deviation from the optimized flow profile is sensed fast enough, the smallness assumption of the initial values is well justified and our scheme will steer the flow back to the desired flow exponentially fast.

In the following section we describe a strategy for the numerical computation of the stabilizing feedback boundary control law based on solving a discretized version of (8). We will see that here, several problems occur as compared to the standard case of discretized operator Riccati equations associated to linear parabolic linear-quadratic optimal control problems. Remedies for those problems are described in Section 3. We provide an outlook on future work in Section 4.

## 2. Numerical Solution of the Operator Riccati Equation

The general problem encountered with the Riccati approach in the context of feedback control for PDEs is that the expected dimensions of the FEM ansatz spaces, called $n$, required to achieve a reasonable approximation error will be far beyond the abilities of ARE solvers known from systems and control theory as the number of unknowns in the Riccati solution is essentially $n^2$. In the following, we will therefore discuss the computational challenges encountered after the spatial FEM discretization is performed.

First note that a Galerkin FEM discretization of (4) (employing the Helmholtz decomposition so that $Pz$ becomes the new state variable) yields a finite-dimensional linear-quadratic optimal control problem for an evolution equation of the form

$$M_h \dot{z}_h(t) = -K_h z_h(t) + B_h u_h(t), \quad z_h(0) = Q_h z_0, \tag{9}$$

where $M_h$, $K_h$ are the mass and stiffness matrices, $B_h$ is the discretized boundary control operator, and $Q_h$ is the canonical projection onto the FEM ansatz space. The stabilizing feedback law for the finite-dimensional approximation to the optimal control problem (6) then becomes

$$u_h(t) = -B_h^T \Pi_h M_h z_h(t) =: -F_h z_h(t), \tag{10}$$

where $\Pi_h = \Pi_h^T$ is the unique nonnegative semidefinite solution to the ARE

$$0 = \mathcal{R}(\Pi_h) = M_h - K_h^T \Pi_h M_h - M_h \Pi_h K_h - M_h \Pi_h B_h B_h^T \Pi_h M_h. \tag{11}$$

This matrix-valued nonlinear system of equations can be considered as a finite-dimensional approximation to the operator Riccati equation (8). Under mild assumptions, it can be proved that for linear parabolic control systems, $\Pi_h$ converges uniformly to $\Pi$ in the strong operator topology, see, e.g., [5, 22, 36, 45, 41, 42]. Convergence rates and a priori error estimates are derived in [40] under mild assumptions for linear parabolic control systems. Note that similar results are not yet available for the situation considered above despite first results in the stationary case [25]. The major computational challenge in this approach is that in order to solve the finite-dimensional problem for computing a feedback law $u_h$ as in (10) for approximating $u$ from (7), we need to solve (11) numerically. As the solution is a symmetric matrix, we are faced with $n(n+1)/2$ unknowns, where $n$ is the dimension of the FEM ansatz space. This is infeasible for 3D problems and even most 2D problems. Even if there were algorithms to handle this complexity, it would in general not be possible to store the solution matrix $\Pi_h$ in main memory or even on a contemporary hard disk. In recent years, several approaches to circumvent this problem have been proposed.

One major ingredient necessary to solve AREs for large-scale systems arising from PDE-constrained optimal control problems is based on the observation that often the eigenvalues of $\Pi_h$ decay to zero rapidly. Thus, $\Pi_h$ can be approximated by a low-rank matrix $Z_h Z_h^T$, where $Z_h \in \mathbb{R}^{n\times r}$, $r \ll n$. Current research focuses on numerical algorithms to compute $Z_h$ directly without ever forming $\Pi_h$. A promising possibility in this direction is to employ Newton's method for AREs. Such methods have been developed in recent years, see [4, 20, 15, 16]. The basis for these methods is the efficient solution of the Lyapunov equations to be solved in each Newton step,

$$(K_h + B_h B_h^T \Pi_{h,j} M_h)^T N_j M_h + M_h N_j (K_h + B_h B_h^T \Pi_{h,j} M_h) - \mathcal{R}(\Pi_{h,j}) = 0, \quad (12)$$

where $\Pi_{h,j}$ are the Newton iterates and $\Pi_{h,j+1} = \Pi_{h,j} + N_j$. It is mandatory to employ the structure in the coefficient matrices: sparse (stiffness, mass) matrix plus low-rank update. This can be achieved utilizing methods based on the ADI iteration [20, 43, 54], so that the computational complexity of the Newton step can be decreased from an infeasible $\mathcal{O}(n^3)$ to merely the complexity of solving the corresponding stationary PDE. Note that the method used in [4] is similar to the one discussed in [20, 15, 16], but the latter one exhibits a better numerical robustness and enhanced efficiency. In the situation considered here, the cost for one Newton step is expected to become proportional to solving the stationary linearized Navier-Stokes problem. This requires some research into the direction of appropriate linear solvers for nonsymmetric saddle point problems, including preconditioners as discussed in [27]. Another issue is that using standard finite-element methods (like for instance the Taylor-Hood elements) for Navier-Stokes equations, the mass matrices in (11) and (12) are singular due to the incompressibility condition. None of the existing solvers for large-scale Lyapunov equations and AREs is able to deal with these issues. Therefore, we first intended to use divergence-free elements computed using explicit Helmholtz projection. A recent new algorithmic idea [35] in this direction inspired a new approach (see Subsection 3.1 below).

Moreover, the efficiency of the discussed approach is much enhanced, when instead of the identity operator $I$, a nonnegative semidefinite operator $W$ or $W_h$ is employed in (8) or (11). This is the case, e.g., if an observation equation is employed: $y = \mathcal{C}z$, where $\mathcal{C}$ is a restriction operator, so that only information localized in some subdomain of $\Omega$ or $\Gamma$ is used in the cost functional. Note that it is quite natural in technical applications that not the complete state is available for optimization and control, but only selected measurements can be employed. Thus, it is reasonable to consider such an observation equation.

Other approaches for large-scale AREs arising in a PDE control context are multigrid approaches [51, 32] and a sign function implementation based on formatted arithmetics for hierarchical matrices proposed in [33]. Both approaches seem not feasible for the matrix structures that will result from the FEM discretization of the linearized Navier-Stokes problems considered here. Also, approaches based on Krylov subspace methods have been investigated for solving large-scale Riccati and Lyapunov equations [18, 37, 38]. These methods are not guaranteed to compute the requested nonnegative semidefinite solution of the ARE and in general are not competitive to the Newton-ADI approach. For these approaches, too, the singularity of the mass matrices is an obstacle and would require deeper investigations itself. Therefore, we will concentrate on the modification of the Newton-ADI approach using the ideas from [35] as this appears to be most promising.

Even though the computational complexity of new methods for large-scale AREs has decreased considerably, the number of columns in $Z_h$ may still be too large in case of 3D problems or if convection is the dominating effect in the flow dynamics. Nonetheless, there are some approaches that allow a direct iteration on the approximate feedback operator $F_h$ [4, 20, 16]. Thus, $Z_h$ need not be computed, and the storage requirements are limited by the number of rows in $F_h$ which equals the number of input variables. It is therefore of vital importance to model the problem such that the number of resulting control variables is as small as possible. If this can be achieved, then the method discussed in [20, 16] is applicable also for 2D and 3D problems while in [4, 40, 51], only 1D problems are considered. Further investigation of the case of a positive definite $W$ and $W_h$ as in the case of $W = I$ is required here. The feedback iteration, too, needs to be adapted in order to be able to deal with the singular mass matrix.

In order to apply our control strategy in practice, we need to find the stabilizing solution of the operator Riccati equation (8). If for the moment we assume that we discretize this equation after the Helmholtz decomposition is performed so that we work with the state variable $Px$ in the set of divergence-free functions, in the end we have to solve numerically the ARE (11). In order to simplify notation, in the following we will write this *Helmholtz-projected ARE* in the form

$$0 = M + (A + \omega I)^T XM + MX(A + \omega I) - MXBB^T XM. \tag{13}$$

Before solving this equation with the now well established low-rank Newton-ADI method, several problems have to be solved:

1. In order to arrive at the matrix representation (13) of (8), the discretization of the Helmholtz-projected Oseen equations (4) would require divergence-free finite elements. As our approach should work with a standard Navier-Stokes solver like NAVIER [6], where Taylor-Hood elements are available, we have to deal in some way with the Helmholtz projector $P$.
2. Each step of the Newton-ADI iteration with $A_0 := A + \omega I$ requires the solution of
$$A_j^T Z_{j+1} Z_{j+1}^T M + M Z_{j+1} Z_{j+1}^T A_j = -W_j W_j^T = -M - M(Z_j Z_j^T B)(Z_j Z_j^T B)^T M,$$
where $n_v := \operatorname{rank} M = $ dim of ansatz space for velocities. This leads to the need to solve $n_v + m$ linear systems of equations in each step of the Newton-ADI iteration, making the approach less efficient.
3. The linearized system associated with $A + \omega I$ is unstable in general. But to start the Newton iteration, a stabilizing initial guess is needed.

In order to deal with the Helmholtz projector, the first strategy employed was to explicitly project the ansatz functions. This requires the solution of one saddle-point problem per ansatz function. Thus, this approach becomes fairly expensive already for coarse-grain discretizations. Nevertheless, we could verify our stabilization strategy using these ideas, first presented in [7]. In particular, with this strategy, the third issue from above is not problematic as several stabilization procedures available for standard linear dynamical systems can be employed. Here, we chose an approach based on the homogeneous counterpart of (13), i.e., the *algebraic Bernoulli equation*

$$0 = (A + \omega I)^T XM + MX(A + \omega I) - MXBB^T XM, \tag{14}$$

which can be solved for the stabilizing solution $X$ efficiently, e.g., with methods described in [12, 13] if the dimension of the problem is not too large.

In order to evaluate the solution strategies, we decided to test the developed numerical algorithms first for the Kármán vortex street. We try to stabilize the flow behind an elliptic obstacle at Re = 500. We use a parabolic inflow profile and a "do-nothing" outflow condition. The uncontrolled flow profile is shown in Figure 1. Clearly, the obstacle produces the expected vortex shedding behind the obstacle.

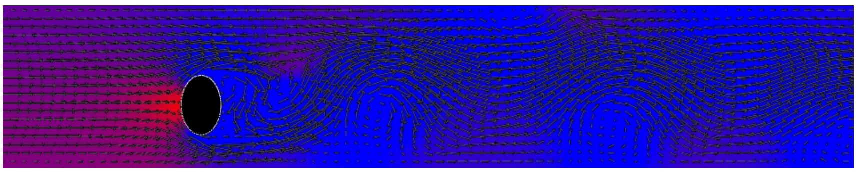


FIGURE 1. Kármán vortex street, uncontrolled.

For testing the strategy described above, we chose $w$ as solution of the stationary Navier-Stokes equations for Re = 1.

As a coarse mesh was employed (approx. 5.000 degrees of freedom for the velocities), the explicit projection of all ansatz functions onto the set of divergence-free functions is possible, so that we arrive at a standard ARE as in (11) that can be solved, e.g., by the method described in [15, 16, 20]. The controlled system using the Dirichlet boundary feedback control approach based on the solution of this ARE is displayed in Figure 2. The figures show the velocity field $v$ of the stabilized flow at various time instances, where the control acts by blowing or sucking at two positions: one at the top of the obstacle, and one at its bottom. Clearly, the desired stabilization of the flow behind the obstacle can be observed.

In the following section, we will describe the developed solution strategies for the encountered problems as described above.

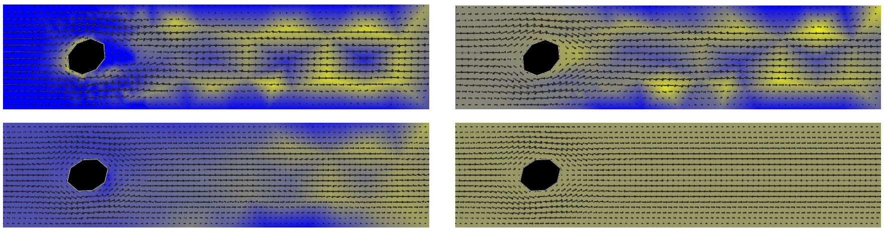


FIGURE 2. Kármán vortex street, controlled, at $t = 1$ (top left), $t = 5$ (top right), $t = 8$ (bottom left), and $t = 10$ (bottom right).

## 3. Remedies of Problems Encountered by the ARE Solver

### 3.1. Avoiding divergence-free FE

A standard FE discretization of the linearized Navier-Stokes system (4) using Taylor-Hood elements yields the following system of differential-algebraic equations (DAEs):

$$\begin{aligned} E_{11}\dot{z}_h(t) &= A_{11}z_h(t) + A_{12}p_h(t) + B_1u(t) \\ 0 &= A_{12}^T z_h(t) + B_2u(t) \\ z_h(0) &= z_{h,0}, \end{aligned} \tag{15}$$

where $E_{11} \in \mathbb{R}^{n_v \times n_v}$ is symmetric positive definite, $A_{12}^T \in \mathbb{R}^{n_p \times n_v}$ is of rank $n_p$.

$z_h \in \mathbb{R}^{n_v}$ and $p_h \in \mathbb{R}^{n_p}$ are the states related tot he FEM discretization of velocities and pressure, and $g \in \mathbb{R}^{n_g}$ the system input derived from the Dirichlet boundary control.

Unfortunately, the ARE corresponding to the DAE (15) in the form (13) (with $M = \mathrm{diag}(E_{11}, 0)$, $A + \omega I = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & 0 \end{bmatrix}$, etc., does not yield the information required about the stabilizing feedback. It is well-known (see, e.g., [44]) that if $M$ is singular, (13) may or may not have solutions, no matter whether the corresponding DAE is stabilizable. Moreover, the positive semidefinite solution of (13), if it exists, may not be stabilizing. In [24] we suggest a projected ARE using spectral projectors onto the deflating subspaces of the matrix pair $(A, M)$ corresponding to the finite eigenvalues. It can then be proved that under standard assumptions on stabilizability of descriptor systems (linear systems in DAE form), the so-obtained ARE has a unique stabilizing positive semidefinite solution as in the regular case of invertible $M$. As a corollary, we can show that the projected version of (13) has a unique stabilizing positive semidefinite solution. Unfortunately, the theory and methods derived in [24] require the explicit availability of the spectral projectors.

Inspired by a similar approach in [35], it turns out that one can avoid the computation of these spectral projectors. First observe that forming these projectors in the form

$$P_h := I_{n_v} - A_{12}(A_{12}^T E_{11}^{-1} A_{12})^{-1},$$

and applying the Newton-ADI iteration to the projected version of the ARE (13), it turns out that in each step of the iteration, we have to solve the Lyapunov equation

$$A_j^T Z_{j+1} Z_{j+1}^T P_h E_{11} P_h^T + P_h E_{11} P_h^T Z_{j+1} Z_{j+1}^T A_j = -W_j W_j^T,$$

where

$$\begin{aligned} A_j &:= P_h(A_{11} - B_1 B_1^T P_h Z_j Z_j^T P_h E_{11}) P_h, \\ W_j &:= \left[ \begin{array}{cc} P_h C^T & P_h E_{11} P_h Z_j Z_j^T P_h B_1 \end{array} \right]. \end{aligned}$$

Thus, a low-rank factor so that $X_{j+1} \approx Z_{j+1} Z_{j+1}^T$ can be computed as

$$Z_{j+1} = \sqrt{\mu} \left[ B_{j,\mu},\, A_{j,\mu} B_{j,\mu},\, A_{j,\mu}^2 B_{j,\mu},\, \ldots,\, A_{j,\mu}^j B_{j,\mu} \right],$$

where $B_{j,\mu}$ solves the saddle point problem

$$\left[ \begin{array}{cc} E_{11} + \mu(A_{11} - B_1 B_1^T Z_j Z_j^T E_{11}) & A_{12} \\ A_{12}^T & 0 \end{array} \right] \left[ \begin{array}{c} B_{j,\mu} \\ * \end{array} \right] = \left[ \begin{array}{cc} C^T & E_{11} Z_j Z_j^T B_1 \\ 0 & 0 \end{array} \right],$$

The multiplication by $A_{j,\mu}$ is realized by the solution of a saddle-point problem with the same coefficient matrix. Hence, the projector $P_h$ needs not be formed explicitly, and we can advance the Newton iteration by solving saddle point problems associated to stationary Oseen-like problems.

### 3.2. Removing $M_h$ from the right-hand side of Lyapunov equations

The solution strategy employed here can be based on a strategy already suggested for standard AREs

$$0 = AX + XA^T - XBB^T X + W. \tag{16}$$

Thus, for simplicity, consider the following Lyapunov equation arising in the Newton step when applying Newton-ADI to (16):

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1} A_j = -W - X_j BB^T X_j \qquad \text{for } j = 1, 2, \ldots$$

By subtracting the two consecutive Lyapunov equations for $j-1, j$ from each other, we obtain

$$A_j^T N_j + N_j A_j = -N_{j-1} BB^T N_{j-1} \qquad \text{for } j = 1, 2, \ldots \tag{17}$$

See [4, 19, 46] for details and applications of this variant. By the subtraction, the constant term $W$ vanishes from the right-hand side of the Lyapunov equation to be solved in the Newton-ADI step. Thus, we are now facing a Lyapunov equation with low-rank right-hand side as desirable for an efficient application of the low-rank ADI method. In particular, this rank equals the number of inputs used for stabilization. In general, this will be fairly low: mostly, $1 \leq m \leq 10$. This strategy can be applied without changes to our situation. Also note that (17) can be written in factored form as required by the low-rank ADI method for Lyapunov equations.

One remaining problem is that in order to start the Newton-ADI iteration based on (17), we need a guess for $N_0 = X_1 - X_0$, i.e., besides a stabilizing inital guess $X_0 = Z_0 Z_0^T$, we also need $X_1 = Z_1 Z_1^T$. Finding $X_0$ is a task on its own (this is actually Problem 3 in the above list and will be discussed in the following subsection).

One possibility here is to compute $X_0$ and $X_1$ from the full, dense Lyapunov equations obtained from a coarse grid discretization of (4), and prolongate these to the fine grid, leading naturally to low-rank factorized forms as required. It remains to investigate the accuracy required for the approximation of $X_1$. Some results in this direction can probably be derived using an interpretation of this approach as inexact Newton method. Recent work on inexact Newton-Kleinman for standard AREs [28] sheds some light on this situation.

A possible refinement of the proposed strategy could involve coarse grid corrections using Richardson iteration or a nested iteration for AREs as in [32].

### 3.3. Computing a stabilizing initial feedback

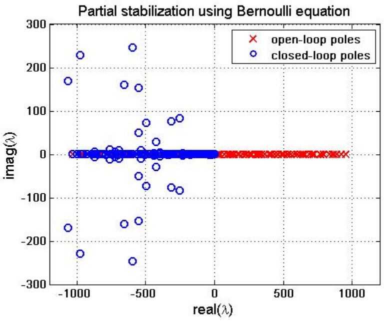
If we use the same notation as in (13), this task can be described as finding a matrix $F_0$ such that all finite eigenvalues of the matrix pair $(A - BB^T F_0, M)$ are contained in the open left half of the complex plane.

There are basically three approaches discussed in the literature to compute such an $F_0$:

- pole placement (see [26] for an overview of algorithms in the standard case that $M$ is invertible, for descriptor systems see [53]);
- Bass-type algorithms based on solving a Lyapunov equation, described for standard systems in [2] while for descriptor systems, see [53, 17, 24];
- $F_0 = B^T X_0$, where $X_0$ is the stabilizing solution of the algebraic Bernoulli equation (14). For standard systems, see [12, 13] for algorithms to compute $X_0$, while descriptor systems are treated in [17].

The Bernoulli stabilization algorithm from [17] was tested so far for a descriptor system obtained from discretizing a Stokes-like equation that can be derived from the linearized Navier-Stokes equations (4) by neglecting the convective terms in (4a). Figure 3 shows that stabilization is achieved even in a case where an artificially strong de-stabilization parameter $\omega = 1000$ is used. This algorithm will be our first choice as stabilization procedure as it turns out to be a lot more robust to the effects of the ill-conditioning of the stabilization problem than all pole placement and Bass-type algorithms; for details see the numerical experiments in [17].

It remains to apply the described strategies in the situation discussed here, i.e., to perturbed flow problems described by the Navier-Stokes system (1). At this writing, the implementation of the ideas described in this section is underway and numerical results obtained with this will be reported elsewhere.



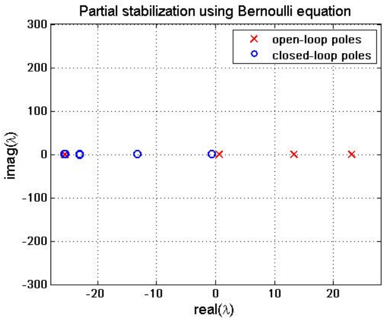




FIGURE 3. Bernoulli stabilization of a Stokes-/Oseen-type descriptor system: shown are the finite eigenvalues of $(A, M)$ (open-loop poles) and of $(A - BB^T F_0, M)$ (closed-loop poles) (left) with a close-up around the origin (right).

## 4. Conclusions and Outlook

We have described a method to stabilize perturbed flow problems, described by the incompressible Navier-Stokes equations, by Dirichlet boundary control. The control may act tangential as well as normal to the boundary. It allows to compensate for disturbances of optimized flow profiles, where an open-loop control may have been obtained by optimization methods. The stabilizing control is of feedback-type so that a closed-loop control system is obtained. The feedback is computed using an associated linear-quadratic optimal control problem for the linearized Navier-Stokes system, projected onto the space of divergence-free functions, arising from linearizing the actual system about a desired stationary solution. The numerical solution of this linear-quadratic optimal control problem is obtained by solving a large-scale algebraic Riccati equation. Several modifications of existing algorithms for this task are necessary in order to achieve an efficient and applicable solution strategy. We have described these modifications here. Their implementation and application to realistic flow control problems is under current investigation.

In the future, we plan to extend the techniques described in this paper to flow problems governed by the incompressible Navier-Stokes equations coupled with field equations describing reactive, diffusive, and convective processes.

Further improvements in the efficiency of the Lyapunov and Riccati solvers using the ADI-Galerkin hybrid method suggested in [21, 23] which often significantly accelerate the ADI and Newton iteration for the Lyapunov and Riccati equations will also be investigated.

## References


[1] O. M. Aamo and M. Krstic. *Flow Control by Feedback. Stabilization and Mixing.* Springer-Verlag, London, 2003.

[2] E. Armstrong. An extension of Bass' algorithm for stabilizing linear continuous constant systems. *IEEE Trans. Automat. Control*, AC–20:153–154, 1975.

[3] M. Badra. Lyapunov function and local feedback boundary stabilization of the Navier-Stokes equations. *SIAM J. Control Optim.*, 48(3):1797–1830, 2009.

[4] H. Banks and K. Ito. A numerical algorithm for optimal feedback gains in high dimensional linear quadratic regulator problems. *SIAM J. Cont. Optim.*, 29(3):499–515, 1991.

[5] H. Banks and K. Kunisch. The linear regulator problem for parabolic systems. *SIAM J. Cont. Optim.*, 22:684–698, 1984.

[6] E. Bänsch. Simulation of instationary, incompressible flows. *Acta Math. Univ. Comenianae*, LXVII(1):101–114, 1998.

[7] E. Bänsch, P. Benner, and A. Heubner. Riccati-based feedback stabilization of flow problems. In *23rd IFIP TC 7 Conference, July 23–27, 2007, Cracow, Poland*, 2007.

[8] V. Barbu. Feedback stabilization of the Navier-Stokes equations. *ESAIM: Control Optim. Calc. Var.*, 9:197–206, 2003.

[9] V. Barbu, I. Lasiecka, and R. Triggiani. Tangential boundary stabilization of Navier-Stokes equations. *Mem. Amer. Math. Soc.*, 181(852):x+128, 2006.

[10] V. Barbu and S. S. Sritharan. $H^\infty$-control theory of fluid dynamics. *Proc. R. Soc. Lond. A*, 454:3009–3033, 1998.

[11] V. Barbu and R. Triggiani. Internal stabilization of Navier-Stokes equations with finite dimensional controllers. *Indiana Univ. Math. J.*, 53(5):1443–1494, 2004.

[12] S. Barrachina, P. Benner, and E. Quintana-Ort. Efficient algorithms for generalized algebraic Bernoulli equations based on the matrix sign function. *Numer. Algorithms*, 46(4):351–368, 2007.

[13] U. Baur and P. Benner. Efficient solution of algebraic Bernoulli equations using H-matrix arithmetic. In K. Kunisch, G. Of, and O. Steinbach, editors, *Numerical Mathematics and Advanced Applications. Proceedings of ENUMATH 2007, the 7th European Conference on Numerical Mathematics and Advanced Applications, Graz, Austria, September 2007*, pages 127–134. Springer-Verlag, Heidelberg, 2008.

[14] P. Benner. Computational methods for linear-quadratic optimization. *Supplemento ai Rendiconti del Circolo Matematico di Palermo, Serie II*, No. 58:21–56, 1999.

[15] P. Benner. Efficient algorithms for large-scale quadratic matrix equations. *Proc. Appl. Math. Mech.*, 1(1):492–495, 2002.

[16] P. Benner. Solving large-scale control problems. *IEEE Control Systems Magazine*, 14(1):44–59, 2004.

[17] P. Benner. Partial stabilization of descriptor systems using spectral projectors. In S. Bhattacharyya, R. Chan, P. Van Dooren, V. Olshevsky, and A. Routray, editors, *Numerical Linear Algebra in Signals, Systems and Control*, Lecture Notes in Electrical Engineering. Springer-Verlag, Berlin/Heidelberg, to appear.

[18] P. Benner and H. Faßbender. An implicitly restarted symplectic Lanczos method for the Hamiltonian eigenvalue problem. *Linear Algebra Appl.*, 263:75–111, 1997.

[19] P. Benner, V. Hernández, and A. Pastor. On the Kleinman iteration for nonstabilizable systems. *Math. Control, Signals, Sys.*, 16:76–93, 2003.

[20] P. Benner, J.-R. Li, and T. Penzl. Numerical solution of large Lyapunov equations, Riccati equations, and linear-quadratic control problems. *Numer. Lin. Alg. Appl.*, 15:755–777, 2008.

[21] P. Benner, R.-C. Li, and N. Truhar. On the ADI method for Sylvester equations. *J. Comput. Appl. Math.*, 233(4):1035–1045, 2009.

[22] P. Benner and J. Saak. Linear-quadratic regulator design for optimal cooling of steel profiles. Technical Report SFB393/05-05, Sonderforschungsbereich 393 *Parallele Numerische Simulation für Physik und Kontinuumsmechanik*, TU Chemnitz, 09107 Chemnitz, FRG, 2005.

[23] P. Benner and J. Saak. A Galerkin-Newton-ADI Method for Solving Large-Scale Algebraic Riccati Equations. Preprint SPP1253–090, DFG Priority Programme *Optimization with Partial Differential Equations*, 2010.

[24] P. Benner and T. Stykel. Numerical algorithms for projected generalized Riccati equations. In preparation.

[25] E. Casas, M. Mateos, and J.-P. Raymond. Error estimates for the numerical approximation of a distributed control problem for the steady-state Navier-Stokes equations. *SIAM J. Control Optim.*, 46(3):952–982, 2007.

[26] B. Datta. *Numerical Methods for Linear Control Systems.* Elsevier Academic Press, 2004.

[27] H. Elman, D. Silvester, and A. Wathen. *Finite Elements and Fast Iterative Solvers: with applications in incompressible fluid dynamics.* Oxford University Press, Oxford, 2005.

[28] F. Feitzinger, T. Hylla, and E. Sachs. Inexact Kleinman-Newton method for Riccati equations. *SIAM J. Matrix Anal. Appl.*, 31:272–288, 2009.

[29] Y. Fernández-Cara, S. Guerrero, O. Imanuvilov, and J.-P. Puel. Local exact controllability of the Navier-Stokes system. *J. Math. Pures Appl., IX. Sér.*, 83(12):1501–1542, 2004.

[30] A. Fursikov. Stabilization for the 3D Navier-Stokes system by feedback boundary control. *Discrete Contin. Dyn. Syst.*, 10(1–2):289–314, 2004.

[31] A. Fursikov, M. Gunzburger, and L. Hou. Optimal boundary control for the evolutionary Navier-Stokes system: The three-dimensional case. *SIAM J. Cont. Optim.*, 43(6):2191–2232, 2005.

[32] L. Grasedyck. Nonlinear multigrid for the solution of large-scale Riccati equations in low-rank and $\mathcal{H}$-matrix format. *Numer. Lin. Alg. Appl.*, 15:779–807, 2008.

[33] L. Grasedyck, W. Hackbusch, and B. Khoromskij. Solution of large scale algebraic matrix Riccati equations by use of hierarchical matrices. *Computing*, 70:121–165, 2003.

[34] M. Gunzburger. *Perspectives in Flow Control and Optimization.* SIAM Publications, Philadelphia, PA, 2003.

[35] M. Heinkenschloss, D. Sorensen, and K. Sun. Balanced truncation model reduction for a class of descriptor systems with application to the oseen equations. *SIAM J. Sci. Comput.*, 30(2):1038–1063, 2008.

[36] K. Ito. Finite-dimensional compensators for infinite-dimensional systems via Galerkin-type approximation. *SIAM J. Cont. Optim.*, 28:1251–1269, 1990.

[37] I. Jaimoukha and E. Kasenally. Krylov subspace methods for solving large Lyapunov equations. *SIAM J. Numer. Anal.*, 31:227–251, 1994.

[38] K. Jbilou and A. Riquet. Projection methods for large Lyapunov matrix equations. *Linear Algebra Appl.*, 415(2–3):344–358, 2006.

[39] B. King and Y.-R. Ou. Nonlinear dynamic compensator design for flow control in a driven cavity. In *Proc. 34th IEEE Conf. Decision Contr.*, pages 3741–3746, 1995.

[40] M. Kroller and K. Kunisch. Convergence rates for the feedback operators arising in the linear quadratic regulator problem governed by parabolic eqautions. *SIAM J. Numer. Anal.*, 28(5):1350–1385, 1991.

[41] I. Lasiecka and R. Triggiani. *Differential and Algebraic Riccati Equations with Application to Boundary/Point Control Problems: Continuous Theory and Approximation Theory.* Number 164 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1991.

[42] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories I. Abstract Parabolic Systems*. Cambridge University Press, Cambridge, UK, 2000.

[43] J.-R. Li and J. White. Low rank solution of Lyapunov equations. *SIAM J. Matrix Anal. Appl.*, 24(1):260–280, 2002.

[44] V. Mehrmann. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution.* Number 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, July 1991.

[45] K. Morris. Design of finite-dimensional controllers for infinite-dimensional systems by approximation. *J. Math. Syst., Estim., and Control*, 4:1–30, 1994.

[46] K. Morris and C. Navasca. Solution of algebraic Riccati equations arising in control of partial differential equations. In *Control and boundary analysis*, volume 240 of *Lect. Notes Pure Appl. Math.*, pages 257–280. Chapman & Hall/CRC, Boca Raton, FL, 2005.

[47] J.-P. Raymond. Boundary feedback stabilization of the incompressible Navier-Stokes equations. Talk at TC7 IFIP Conference, Torino, July 18-22, 2005.

[48] J.-P. Raymond. Local boundary feedback stabilization of the Navier-Stokes equations. In *Control Systems: Theory, Numerics and Applications, Rome, 30 March – 1 April 2005*, Proceedings of Science. SISSA, 2005.

[49] J.-P. Raymond. Feedback boundary stabilization of the two-dimensional Navier-Stokes equations. *SIAM J. Control Optim.*, 45(3):790–828, 2006.

[50] J.-P. Raymond. Feedback boundary stabilization of the three-dimensional incompressible Navier-Stokes equations. *J. Math. Pures Appl. (9)*, 87(6):627–669, 2007.

[51] I. Rosen and C. Wang. A multi–level technique for the approximate solution of operator Lyapunov and algebraic Riccati equations. *SIAM J. Numer. Anal.*, 32(2):514–541, 1995.

[52] V. Sima. *Algorithms for Linear-Quadratic Optimization*, volume 200 of *Pure and Applied Mathematics.* Marcel Dekker, Inc., New York, NY, 1996.

[53] A. Varga. On stabilization methods of descriptor systems. *Sys. Control Lett.*, 24:133–138, 1995.

[54] E. Wachspress. Iterative solution of the Lyapunov matrix equation. *Appl. Math. Letters*, 107:87–90, 1988.

Eberhard Bänsch
FAU Erlangen
Lehrstuhl Angewandte Mathematik III
Haberstr. 2
91058 Erlangen
Germany
e-mail: `baensch@am.uni-erlangen.de`

Peter Benner
Max Planck Institute for Dynamics of Complex Technical Systems
Sandtorstr. 1
39106 Magdeburg
Germany
e-mail: `benner@mpi-magdeburg.mpg.de`