Second Order to Second Order Balancing for Index-1 Vibrational Systems

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***Abstract*—Linearizing constraint equations of motion around equilibrium points in mechanics or coupling electrical and mechanical parts in mechatronics one obtains large sparse second-order index-1 differential algebraic (DAE) models. To get reduced order models of such systems, first they can be rewritten into first-order models. Then, model reduction techniques are applied to these first order representations to get a reduced first-order index-1 or standard state space model. Unfortunately, it is not possible to go back to second order formulation from these reduced systems, though it is often desirable to work with second order surrogate models. In this paper, we present algorithms to retrieve reduced index-1 DAE or standard second order ODE systems and apply these to a micro-mechanical piezo-actuated structural FEM model of a certain building block of a machine tool. Numerical results illustrate the efficiency of the techniques.**

***Index Terms*—piezo actuator, second order index-1 systems, model reduction, balanced truncation, ADI-methods.**

I. Introduction

The model that we consider throughout this paper has the following form:

$$\begin{aligned} M\ddot{z}(t) + D\dot{z}(t) + Kz(t) &= Hu(t), \\ y(t) &= Lz(t), \end{aligned} \tag{1}$$

where $M, D, K \in \mathbb{R}^{n\times n}$ are the sparse FEM-matrices resulting from the modeling and known as mass, damping and stiffness matrices respectively. $H \in \mathbb{R}^{n\times p}$ is the input matrix describing the external access to the system and $L \in \mathbb{R}^{m\times n}$ represents the measurements. Correspondingly, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^m$ are the control input to the system and the measured output. M, D, K, B and C respectively, have the following structures:

$$M = \begin{bmatrix} M_{vv} & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} D_{vv} & 0 \\ 0 & 0 \end{bmatrix}, \quad K = \begin{bmatrix} K_{vv} & K_{v\Phi} \\ K_{v\Phi}^T & K_{\Phi\Phi} \end{bmatrix},$$
$$H = \begin{bmatrix} B_v \\ B_\Phi \end{bmatrix}, \quad \text{and} \quad L = \begin{bmatrix} C_v & C_\Phi \end{bmatrix},$$

where $M_{vv}, D_{vv}, K_{vv} \in \mathbb{R}^{n_d\times n_d}$ and $K_{\Phi\Phi} \in \mathbb{R}^{n_a\times n_a}$ is always invertible. The first order state space representations of such systems turn out to be index-1 descriptor systems [1] (see also [2]). Therefore, we define such systems as second order index-1 differential algebraic systems. In the special case of the piezo-mechanical system used in the numerical experiments $z = (v^T, \Phi^T)^T$, where v is the vector of mechanical displacements and Φ is the vector of electrical potentials [3]. Eliminating the algebraic parts from (1) employing Schur complements, one can put this into the following compact form

$$\begin{aligned} \mathcal{M}\ddot{v}(t) + \mathcal{D}\dot{v}(t) + \mathcal{K}v(t) &= \mathcal{B}u(t), \\ y(t) &= \mathcal{C}v(t) + \mathcal{D}_a u(t), \end{aligned} \tag{2}$$

where $\mathcal{M} = M_{vv}, \mathcal{D} = D_{vv}, \mathcal{K} = K_{vv} - K_{v\Phi}K_{\Phi\Phi}^{-1}k_{v\Phi}^T$, $\mathcal{B} = B_v - K_{v\Phi}K_{\Phi\Phi}^{-1}B_\Phi, \mathcal{C} = C_v - C_\Phi K_{\Phi\Phi}^{-1}k_{v\Phi}^T$, and $\mathcal{D}_a = C_\Phi K_{\Phi\Phi}^{-1}B_\Phi$.
Equation (1) and (2) are equivalent, because both have the same finite eigenvalues and they are different realizations [4] of the transfer function matrix defined by

$$G(s) = \mathcal{C}(\mathcal{M}s^2 + \mathcal{D}s + \mathcal{K})^{-1}\mathcal{B} + \mathcal{D}_a, \tag{3}$$

with $s \in \mathbb{C}$. $G(s)$ maps the inputs to the outputs of the system in frequency domain. However, computing the $\mathcal{K}$, $\mathcal{B}$ and $\mathcal{C}$ matrices explicitly inverting $K_{\Phi\Phi}$ makes the previously sparse data from the FEM modeling dense [5] and should be avoided in practice.

We want to approximate (2) by a much lower dimensional surrogate system

$$\begin{aligned} \hat{\mathcal{M}}\ddot{\hat{v}}(t) + \hat{\mathcal{D}}\dot{\hat{v}}(t) + \hat{\mathcal{K}}\hat{v}(t) &= \hat{\mathcal{B}}u(t), \\ \hat{y}(t) &= \hat{\mathcal{C}}\hat{v}(t) + \mathcal{D}_a u(t), \end{aligned} \tag{4}$$

where $\hat{\mathcal{M}}$ is invertible, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}} \in \mathbb{R}^{\ell\times\ell}$, $\mathcal{B} \in \mathbb{R}^{\ell\times p}$, $\mathcal{C} \in \mathbb{R}^{m\times\ell}$ and $\ell \ll n$. The reduced order model here is supposed to fulfill some certain approximation requirements, for instance the approximation error $\|y(t) - \hat{y}(t)\|$, or correspondingly $\|G(.) - \hat{G}(.)\|$, (where $\hat{G}(.)$ is the transfer function matrix of the reduced model) should be small in some suitable norm, e.g., the $\mathcal{H}_\infty$ or $\mathcal{H}_2$ norms (see [4]).

The methods to form balanced truncation based reduced order models (4) from (2) are already discussed in [6], [7], [8]. Unfortunately it is prohibitive to form the representation (2) from (1) due to memory restriction and computational cost, since it becomes densely populated, as mentioned above.

The main contribution of this paper is to form (2) only implicitly and work on the sparse formulation of the original model (1). Moreover, a small algebraic manipulation of (4) turns out a reduced second order index-1 model if that is desired.

II. Balanced Truncation

Balanced truncation [4] is a technique to find a balanced system realization, where the system controllability and observability Gramians are equal and diagonalized. The diagonal elements, which are decreasingly ordered, are the system's Hankel singular values. When a system is balanced it can easily be identified via the Hankel singular values which

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states contribute only little in passing energy in a system [4]. Therefore forming a reduced order model by truncating singular values, i.e., less important states, is sufficient from an energetic point of view.

The main ingredient in the derivation of the truncating projection matrices for forming the reduced order model in balanced truncation is to compute the controllability Gramian (P) and observability Gramian (Q), which are the solutions of the two generalized continuous time algebraic Lyapunov equations

$$\begin{array}{rcrcl} APE^T & + & EPA^T & = & -BB^T \quad \text{and} \\ A^TQE & + & E^TQA & = & -C^TC, \end{array} \tag{5}$$

respectively, corresponding to the generalized system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + D_a u(t). \tag{6}$$

The system in (2) can always be transformed into the form (6). There are however multiple choices to form E, A, B and C [5], all having their advantages and disadvantages. The most common practice is as follows:

$$E = \begin{bmatrix} 0 & \mathcal{F} \\ \mathcal{M} & \mathcal{D} \end{bmatrix}, \quad A = \begin{bmatrix} \mathcal{F} & 0 \\ 0 & -\mathcal{K} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \mathcal{B} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \mathcal{C} \end{bmatrix}, \quad \text{and} \quad D_a = -\mathcal{D}_a, \tag{7}$$

where $\mathcal{F}$ can be any invertible matrix. For example, choosing $\mathcal{F}$ as the identity matrix (I) is the simplest or $\mathcal{F} = \mathcal{M}$ makes the system matrices in (6) symmetric when $\mathcal{M}$, $\mathcal{D}$ and $\mathcal{K}$ are symmetric. We assume that the original model (1) is asymptotically stable, i.e., all finite eigenvalues of the model lie in the open left half plane [9]. Then P and Q are the unique positive semi-definite solutions of the above two Lyapunov equations and will be block subdivided as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

according to the block sizes in E and A.

Due to the fact, that the new state variable in this phase space representation is formed by stacking positions (or displacements) on top of velocities (time derivatives of the displacements), we denote the sub-Gramians P_{11}, P_{22} as position and velocity controllability Gramians and analogously Q_{11}, Q_{22} are called position and velocity observability Gramians [7].

The classical algorithm to perform balanced truncation is the square root method which can be characterized in the following few steps

Step 1 Compute the Cholesky factors of P and Q such that

$$P = SS^T \quad \text{and} \quad Q = RR^T. \tag{8}$$

Step 2 Compute the singular value decomposition

$$R^TES = U\Sigma V^T = \begin{bmatrix} U_1, U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}. \tag{9}$$

Step 3 Then the balancing truncation operators are defined as

$$T_l := RU_1\Sigma_1^{-\frac{1}{2}}, \qquad T_r := SV_1\Sigma_1^{-\frac{1}{2}}. \tag{10}$$

Step 4 Finally, the reduced order model is computed by pre- and post-multiplication of the system matrices in (6) with T_l^T and T_r, respectively.

The same procedure can be applied to find a second order reduced order model by balancing the system applying position-position, velocity-velocity, position-velocity or velocity-position balancing (see [7] for further details), which simply replaces the full Gramians by the corresponding combination of position and velocity Gramians in the above procedure. The reduced order model (4) is then formed in the fourth step by computing the low dimensional system matrices $\hat{\mathcal{M}}$, $\hat{\mathcal{D}}$, $\hat{\mathcal{K}}$, $\hat{\mathcal{B}}$, and $\hat{\mathcal{C}}$ as:

$$\begin{aligned} &\hat{\mathcal{M}} = T_l^T M_{vv} T_r, \quad \hat{\mathcal{D}} = T_l^T D_{vv} T_r, \\ &\hat{\mathcal{K}} = \hat{K}_{vv} - \hat{K}_{v\Phi} K_{\Phi\Phi}^{-1} \hat{K}_{v\Phi}^T, \\ &\hat{\mathcal{B}} = \hat{B}_v - \hat{K}_{v\Phi} K_{\Phi\Phi}^{-1} B_\Phi, \quad \hat{\mathcal{C}} = \hat{C}_v - C_\Phi K_{\Phi\Phi}^{-1} \hat{K}_{v\Phi}^T, \end{aligned} \tag{11}$$

where

$$\hat{K}_{vv} = T_l^T K_{vv} T_r, \quad \hat{K}_{v\Phi} = T_l^T K_{v\Phi}, \quad \hat{K}_{v\Phi}^T = K_{v\Phi}^T T_r,$$
$$\hat{B}_v = T_l^T B_v, \quad \hat{C}_v = C_v T_r.$$

A simple algebraic manipulation gives a reduced second order index-1 system

$$\begin{aligned} \hat{M}\ddot{\hat{z}}(t) + \hat{D}\dot{\hat{z}}(t) + \hat{K}\hat{z}(t) &= \hat{H}u(t), \\ \hat{y}(t) &= \hat{L}\hat{z}(t), \end{aligned} \tag{12}$$

with

$$\hat{M} = \begin{bmatrix} \hat{\mathcal{M}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{D} = \begin{bmatrix} \hat{\mathcal{D}} & 0 \\ 0 & 0 \end{bmatrix}, \quad \hat{K} = \begin{bmatrix} \hat{K}_{vv} & \hat{K}_{v\Phi} \\ \hat{K}_{v\Phi}^T & K_{\Phi\Phi} \end{bmatrix},$$
$$\hat{H} = \begin{bmatrix} \hat{B}_v \\ B_\Phi \end{bmatrix}, \quad \text{and} \quad \hat{L} = \begin{bmatrix} \hat{C}_v & C_\Phi \end{bmatrix},$$

and $\hat{\mathcal{M}}, \hat{\mathcal{D}}, \hat{K}_{vv}, \hat{K}_{v\Phi}, \hat{K}_{v\Phi}^T, \hat{B}_1$, and $\hat{C}_1$ obtained as above.

Note that in the large scale setting we are interested in here, the triangular Cholesky factors in Step 1 are replaced by rectangular low rank Cholesky factors which are computed successively columnwise using, e.g., a low rank alternating directions implicit (ADI) iteration as described in the next section. Note further that having computed the factors of the full Gramians, we can easily determine factors for the position and velocity Gramians from the correspondingly sized upper and lower blocks in S and R, i.e., $P_{11} = S_1S_1^T$, $P_{22} = S_2S_2^T$, $Q_{11} = R_1R_1^T$, $Q_{22} = R_2R_2^T$, where

$$S = \begin{bmatrix} S_1 \\ S_2 \end{bmatrix}, \qquad R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

III. Solution of Lyapunov Equations

A. Generalized Sparse Low-Rank Cholesky Factor (GSLRCF) ADI-methods

In the previous sections we have seen that the main ingredients of balancing a system are the two factors of two Gramians that can be found by solving two Lyapunov equations. For a large scale problem, the ADI-method [10] is our preferred solver to compute low rank factors of the two Gramians. In [8] the authors introduced efficient algorithms solving such Lyapunov equations for the ODE systems like (2), where the matrices $\mathcal{M}$, $\mathcal{D}$, $\mathcal{K}$ are sparse. Recently, these methods have been extended to the case of index-1 DAE systems [8], [5], [1] where one is interested in a first order standard state space reduced order model. There, the ideas in [11] are exploited to form the system (2) as present here only implicitly.

The key idea for efficient computations in all cases is the exploitation of the block structure of the matrices in (7). Doing

Algorithm 1 Second order GSLRCF-ADI methods

Input: $\mathcal{M}, \mathcal{D}, \mathcal{K}, \mathcal{B}$ and shift parameters $\{\mu_1, \mu_2, \cdots, \mu_J\}$.
Output: S_1 and S_2.

1: Solve $(\mu_1^2\mathcal{M} - \mu_1\mathcal{D} + \mathcal{K})x_2^1 = -\mathcal{B}$.
2: $S_2^1 = V_2^1 = \sqrt{-2\mu_1}x_2^1$ and $S_1^1 = V_1^1 = -\mu_1 V_2^1$.
3: **for** $i = 2 : i_{max}$ **do**
4: Solve
$(\mu_i^2\mathcal{M} - \mu_i\mathcal{D} + \mathcal{K})x_2^i = (\mu_i\mathcal{M} - \mathcal{D})V_2^{i-1} - \mathcal{M}V_1^{i-1}$
5: $x_1^i = V_2^{i-1} - \mu_i x_2^i$
6: $V_1^i = \sqrt{\frac{Re(\mu_i)}{Re(\mu_{i-1})}}(V_1^{i-1} - (\mu_i + \mu_{i-1}^*)x_1^i)$,
7: $V_2^i = \sqrt{\frac{Re(\mu_i)}{Re(\mu_{i-1})}}(V_2^{i-1} - (\mu_i + \mu_{i-1}^*)x_2^i)$.
8: $S_1^i = \begin{bmatrix} S_1^{i-1} & V_1^i \end{bmatrix}$ and $S_2^i = \begin{bmatrix} S_2^{i-1} & V_2^i \end{bmatrix}$.
9: **end for**

so, the low rank ADI results in Algorithm 1 for computing the observability Gramians.

Applying the algorithm to $\mathcal{M}^T$, $\mathcal{D}^T$, $\mathcal{K}^T$, $\mathcal{C}^T$ results in the computation of the observability Gramians R_1, R_2 and thus R.

In the previous section we have seen that $\mathcal{K}$, $\mathcal{B}$ and $\mathcal{C}$ are dense. As a result, solving a linear system in each iteration step in Algorithm 1 is prohibitively expensive. Therefore, instead of solving such a linear system in each iteration step, following [1] we solve the following linear system:

$$\begin{bmatrix} \mu_i^2 M_{vv} - \mu_i D_{vv} + K_{vv} & K_{v\Phi} \\ K_{v\Phi}^T & K_{\Phi\Phi} \end{bmatrix} \begin{bmatrix} x_2^i \\ \Lambda \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}. \quad (13)$$

where $f_1 = B_v$ and $f_2 = B_\Phi$ for $i = 1$, or $f_1 = (\mu_i\mathcal{M} - \mathcal{D})V_2^{i-1} - \mathcal{M}V_1^{i-1}$ and $f_2 = 0$, otherwise. Although the dimension of this system is higher, the solution is computed much faster since one can exploit the sparse matrices.

B. ADI-shift Parameter Computation

For fast convergence of GSLRCF ADI-methods, a proper set of shifts plays a crucial role. Here we consider the heuristic shift computation procedure going back to [12].

In this procedure first one approximates a subset of the spectrum $\mathcal{S} = \{\mathcal{S}_+, \mathcal{S}_-\}$ by computing a number of large magnitude Ritz-values $\mathcal{S}_+$ (k_+ many) and a number of small magnitude Ritz-values $\mathcal{S}_-$ (k_- many) for the system of interest. In the case of (1) we clearly need to restrict ourselves to the finite spectrum, such that in fact we better use the form (2) in the Ritz value computation. Since this is performed by an Arnoldi method again we never need to form the matrices explicitly, but can work with functions implementing the application by successive propagation through the matrix, i.e., especially we need not form the inverse, but perform a linear system solve there. Then the shifts μ_j for Algorithm 1 are computed from $\mathcal{S}$ by solving the restricted ADI mini-max problem

$$\min_{\mu_1,\mu_2,\cdots,\mu_J} \max_{\lambda\in\mathcal{S}} \prod_{i=1}^{J} \frac{|\,\mu_i - \lambda\,|}{|\,\mu_i + \lambda\,|}. \quad (14)$$

For more details see, e.g, [12].

IV. Numerical Results

The accuracy and performance of the proposed technique, is illustrated by applying to a set of data for the finite element discretization of an adaptive spindle support (ASS) [3] with 290 137 degrees of freedom.

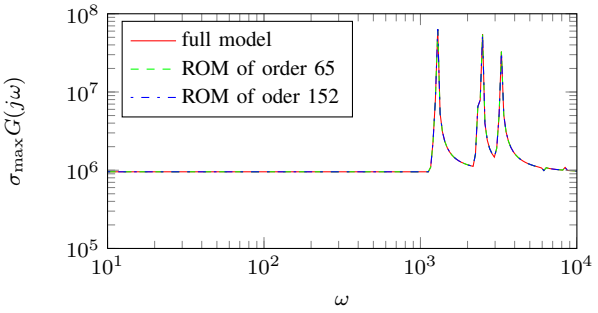

Fig. 1. Sigma plot (maximum singular values) of full and reduced order models.

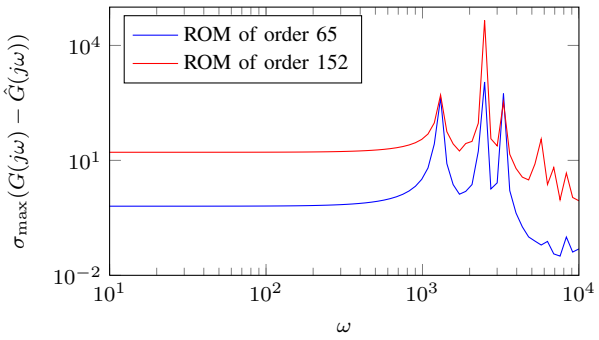

Fig. 2. Absolute error in the sigma plot of full and reduced order models.

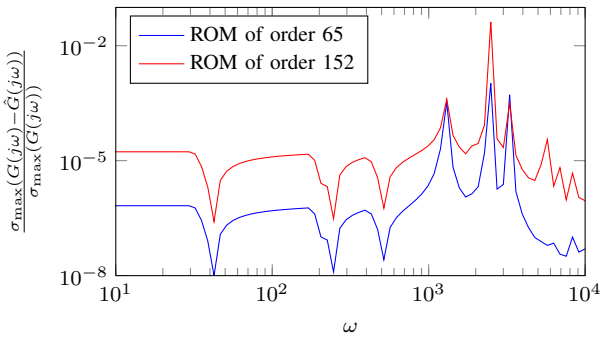

Fig. 3. Relative error in the sigma plot of full and reduced order models.

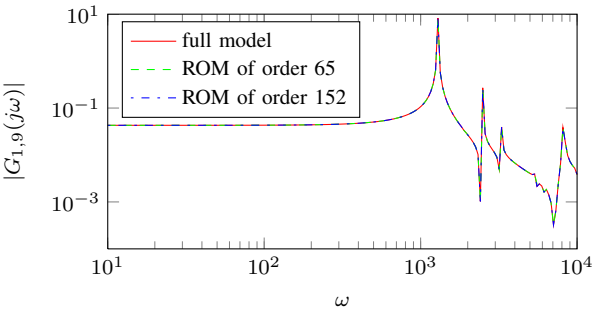

Fig. 4. Frequency responses for input 9 to output 1 relation of full and reduced models

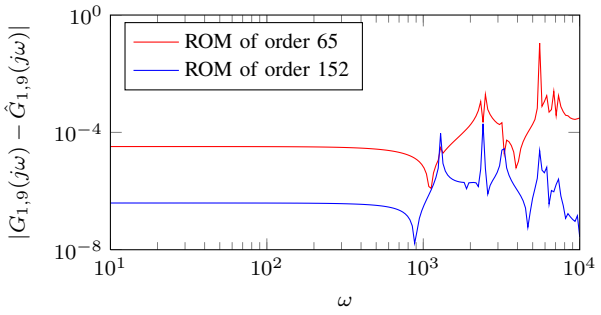

Fig. 5. Absolute error in frequency responses for input 9 to output 1

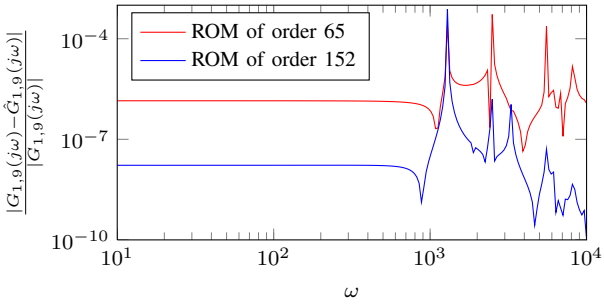

Fig. 6. Relative error in frequency responses for input 9 to output 1

All results have been obtained by using MATLAB®7.11.0 (R2010b) on a board with 4 Intel® Xeon® E7-8837 CPUs with a 2.67-GHz clock speed, 8 Cores each and 1TB of total RAM. An amount of 8-16GB of RAM (depending on the desired accuracy) should however be sufficient for the execution easily. We compute 50 ADI-shift parameters out of $k_+ = 80$ large and $k_- = 70$ small magnitude Ritz values using Penzl's heuristic procedure mentioned above. Then compute the low-rank factors of both position and velocity controllability and observability Gramians applying the algorithm discussed in Section III-A. In the computation of the low-rank Gramian factors, we restrict ourselves to a maximum of 300 iteration steps in both cases to limit the memory consumption.

Following the procedure in (9) – (11) we form exemplary reduced order models of dimension 152 and 65 respectively via position-velocity balancing. The overall computation including the computation of the reduced order model, as well as the numerical error analysis (which is only needed for the presentation of the accuracy here) takes roughly 6-8 hours depending on the desired accuracy again. Here about half of the time goes into the error analysis stage.

Figure 1 shows the frequency responses (largest singular value of $G(\omega)$) of full and both the 152 and the 65 dimensional reduced order models in frequency domain over the frequency (ω) range of 10^1 to 10^4. In Figure 2, the absolute error between the frequency responses of full and reduced (152 and 65) models are shown. The relative errors of both lower dimensional models from full models are depicted in Figure 3.

Consider the single input single output relations for example input 9 (electric potential at one of the actuators) to output 1 (mechanical displacement), Figure 4 shows the frequency responses of both lower dimensional models nicely match the full model. Figures 5 and 6 respectively, show absolute and relative deviation between full and both reduced systems. Analyzing the errors in all figures we have seen that the 152 dimensional reduced order model is better than 65 dimensional one, but both show relative errors far below the percent range and should perform well in ROM-based controller design.

V. Conclusions

We have presented a second order to second order balancing strategy for second order index-1 differential algebraic systems arising, e.g., in the modeling of piezo actuated vibrational models. We extended earlier work, where a first order reduced order model was created. As in that case also our new strategy allows to work on the original FEM matrices and exploit their sparsity in the solver. The efficiency and accuracy has been demonstrated for one large FEM model of an adaptive spindle support employing piezo actuators with almost 300 000 degrees of freedom, proving the applicability of our method in real world applications.

VI. Acknowledgments

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