A Robust Algorithm for Parametric Model Order Reduction

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A robust algorithm for computing reduced-order models of parametric systems is proposed. Theoretical considerations suggest that our algorithm is more robust than previous algorithms based on explicit multi-moment matching. Moreover, numerical simulation results show that the proposed algorithm yields more accurate approximations than traditional non-parametric model reduction methods and parametric model reduction methods based on explicitly computing moments.

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1 Introduction

Many physical models of engineering problems contain some parameters. Here, we consider parametric systems in the form

$$C(p_1, p_2, \ldots, p_m)\dot{x}(t) = G(p_1, p_2, \ldots, p_m)x(t) + Bu(t), \qquad y(t) = L^{\mathrm{T}}x(t). \tag{1}$$

Here, $x(t) \in \mathbf{R}^n$ is the state of the system, $u(t) \in \mathbf{R}^m$ and $y(t) \in \mathbf{R}^q$ are inputs and outputs, respectively, and C, G, B, L are (parameter-dependent) matrices of appropriate sizes. The parameters $p_1, p_2, \ldots, p_\ell$ can be used to describe for example geometry and/or material variations in the physical system and usually some of them will vary from simulation to simulation.

We are interested in parametric systems of very large order n, so that repeated simulation runs are too time consuming for the user. Classical model order reduction techniques [1] for non-parametric system are well known to overcome this difficulty in many situations. At present, few methods [2, 3, 4] can deal with parametric systems as in (1) in the sense that the parameters are preserved in the reduced-order model. These methods are called *parametric model order reduction (PMOR) methods*. Using parametric reduced-order model, one can hope to be able to do perform many simulations with different parameter values with acceptable accuracy and in reasonable time.

This paper is based on the idea proposed in [2], which can be considered as an extension of explicit moment matching (see, e.g., [1]) to PMOR. In [2], certain vectors associated with the moments (in the following text, we call them moment vectors) of the parametric system are computed to construct a projection matrix V, which is then used for computing the reduced-order model. As in the non-parametric case, there are numerical difficulties with the construction of V based on explicitly computing the moment vectors as the computed moment vectors tend to become linearly dependent rather quickly. Even if these vectors are orthogonalized, the computed projection matrix V will in general not lead to an accurate reduced-order parametric model. In the following section, we propose a robust numerical algorithm for realizing the method from [2].

2 An implicit-moment matching PMOR algorithm

The proposed algorithm computes the moment vectors implicitly rather than explicitly. The method can be interpreted as an Arnoldi-type method using a modified Gram-Schmidt implementation. In this way, good numerical stability properties can be expected as an orthonormal basis of the projection subspace (which is needed for computing accurate projectors for model order reduction) can be obtained. The proposed algorithm can deal with both single-input and multi-input linear parametric systems without limitation on the number of parameters in the system.

Due to space limitations, we only give a sketch of the algorithm for a one parameter system ($\ell = 1$) with $C(p_1) \equiv C$ constant and $G(p_1) \doteq G(p) = G_0 + pG$. (A detailed description for the general case can be found in [5], a similar approach is pursued in [3].) If we apply the Laplace transformation to (1), we have the following expression for the state vector $X(s)$ in frequency domain (assuming G_0 to be nonsingular and a certain smallness of s, p):

$$\begin{aligned} X(s) &= (sC - G_0 - pG)^{-1}BU(s) = [I - (sG_0^{-1}C - pG_0^{-1}G)]^{-1}(-G_0)^{-1}BU(s) \\ &= \sum_{i=0}^{\infty}(sG_0^{-1}C - pG_0^{-1}G)^i(-G_0)^{-1}BU(s) \\ &= \{I + (sM_s + pM_p)B_M + [s^2M_s^2 + sp(M_sM_p + M_pM_s) + p^2M_p^2]B_M + \ldots\}U(s), \end{aligned} \tag{2}$$

where $M_s = G_0^{-1}C, M_p = -G_0^{-1}G, B_M = -G_0^{-1}B$. The moment vectors of the parametric system are the coefficients in the above series expansion, i.e. $B_M, M_sB_M, M_pB_M, M_s^2B_M, M_p^2B_M, (M_sM_p + M_pM_s)B_M, M_s^3B_M, \ldots$. From the

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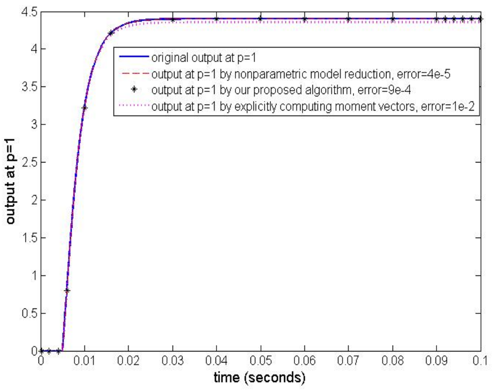

Fig. 1 comparison of outputs by different MOR methods

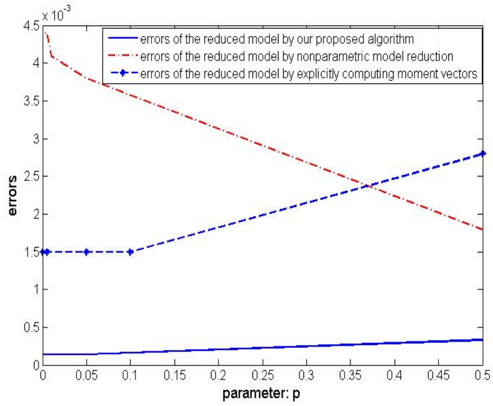

Fig. 2 errors of the outputs computed by different MOR methods at different values of the parameters

second line of (2), $X(s) = \sum_{i=0}^{\infty}(sM_s + pM_p)^i B_M U(s)$, we see that the moment vectors satisfy the recursive relations

$$R_0 = B_M,\ R_1 = [M_sR_0, M_pR_0],\ R_2 = [M_sR_1, M_pR_1],\ \ldots,\ R_j = [M_sR_{j-1}, M_pR_{j-1}]\ldots. \tag{3}$$

where R_j corresponds to the moment vectors in the jth term of the above series expansion in (2). Instead of explicitly computing the moment vectors, we make use of the above recursion, and combine it with the Gram-Schmidt method to implicitly compute the moment vectors so that each new computed vector is immediately orthogonalized against its predecessors, including possible deflation.

3 Numerical simulation results

In the following, we give simulation results for a parametric system describing an anemometer. The system is in the simple form used in the previous section, i.e., C is constant and $G(p_1, p_2, \ldots, p_\ell) \equiv G_0 + pG$. The parameter satisfies $p \in [0, 1]$. The output response of the system is the temperature difference between the sensors on the left and right of the anemometer. We use the step function as input signal $u(t)$. The order of the system is $n = 29,008$.

In Figures 1,2, we compare the proposed PMOR algorithm with traditional non-parametric model reduction technique using a one-sided Arnoldi method. We also include simulation results obtained when explicitly computing the moment vectors. For the Arnoldi method, we have to fix p (here, we choose $p = 1$) to compute the projection matrix V. The reduced-order models obtained from all the three methods are of order $r = 10$. The errors reported in Figure 2 show the difference between the outputs of the reduced-order models and the original model.

In Figure 1, we show the output responses of the three different reduced-order models obtained for $p = 1$. We can see that both our proposed method and the Arnoldi method are very accurate, the PMOR method based on explicit moment matching has a larger error, confirming the better numerical robustness of our approach. Of course, as we use the same parameter value as used for generating the reduced-order model with the non-parametric Arnoldi approach, we can not expect our method to be better in this situation.

In Figure 2, we show the errors of the output responses computed by three different model reduction methods at other values of the parameter p. We see that the new algorithm is most accurate, and this time, the non-parametric model reduction method exhibits the largest errors as it can be expected to be only good in the vicinity of $p = 1$. The PMOR method from [2] behaves a little better than the non-parametric method for small p. However, our new PMOR algorithm is much more accurate over the full parameter range due to the better numerical robustness obtained by implicitly computing the moment vectors.

The above simulation results show that PMOR is necessary for model reduction of parametric systems. The proposed algorithm for implicit computation of the moment vectors effectively improves the numerical accuracy of the reduced parametric model (which is mathematically equivalent to the one obtained in [2]).

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