Balanced Truncation for Stochastic Linear Systems with Guaranteed Error Bound

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***Abstract*— We consider model order reduction of stochastic linear systems by balanced truncation. Two types of Gramians are suggested, both satisfying generalized Lyapunov equations. The first is motivated by energy functionals, the second is taylored to yield an error bound for the truncated system.**

***Index Terms*— Stochastic linear system, balanced truncation, H^∞-error bound, *MSC2010:* 93E03, 93B11, 93B36**

I. INTRODUCTION

Model order reduction by balanced truncation is a standard method, which has been introduced in [9], [11] for linear deterministic control systems of the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad \sigma(A) \subset \mathbb{C}_- \ , \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$, $u(t) \in \mathbb{R}^m$ are the state, output and input of the system, respectively, while $\sigma(A)$ is the spectrum of A. Balanced truncation preserves asymptotic stability and provides guaranteed error bounds. The balancing transformation is computed from the Gramians P and Q, which solve the dual Lyapunov equations

$$AP + PA^T = -BB^T, \quad A^TQ + QA = -C^TC \ . \tag{2}$$

These equations are essential in the characterization of stability, controllability and observability of system (1).

In the case of an asymptotically mean-square stable stochastic linear systems of Itô-type,

$$dx = Ax\,dt + Nx\,dw + Bu\,dt \ , \quad y = Cx \ , \tag{3}$$

the Lyapunov equations typically have to be replaced by the generalized Lyapunov equations (e.g. [8], [7], [4]),

$$AP + PA^T + NPN^T = -BB^T \ , \tag{4}$$

$$A^TQ + QA + N^TQN = -C^TC \ , \tag{5}$$

where A, B, C are as in (1) and $N \in \mathbb{R}^{n\times n}$. Therefore it seems natural that P and Q can be interpreted as Gramians of stochastic systems and that the method of balanced truncation can be carried over. This has partly been worked out in [3] and will be summarized in this note.

However, there are some open questions concerning the existence of error bounds for the reduced system. Here, as an alternative, we present a modified pair of Gramians which leads to an H^∞-type error bound of the same form as for deterministic systems.

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II. GRAMIANS BASED ON ENERGY FUNCTIONALS

Consider a stochastic linear control system of Itô-type

$$dx = Ax\,dt + Nx\,dw + Bu\,dt \ , \quad y = Cx \ , \tag{6}$$

where $w = (w(t))_{t\in\mathbb{R}_+}$ is a zero mean real Wiener process on a probability space $(\Omega, \mathcal{F}, \mu)$ with respect to an increasing family $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ (e.g. [2], [10]).
Let $L^2_w(\mathbb{R}_+, \mathbb{R}^q)$ denote the corresponding space of non-anticipating stochastic processes v with values in $\mathbb{R}^q$ and norm

$$\|v(\cdot)\|^2_{L^2_w} := \mathcal{E}\left(\int_0^\infty \|v(t)\|^2 dt\right) < \infty,$$

where $\mathcal{E}$ denotes expectation. We assume that the homogeneous equation $dx = Ax\,dt + Nx\,dw$ is asymptotically mean-square-stable, i.e. $\mathcal{E}(\|x(t)\|^2) \overset{t\to\infty}{\longrightarrow} 0$, for all initial conditions $x(0) = x_0$. Its fundamental solution will be denoted by Φ, so that $x(t) = \Phi(t, 0)x_0$. Since stochastic differential equations in general can only be solved forward in time (e.g. [10]), note that $\Phi(t, \tau)$ is only defined for $t \geq \tau$. For $t \geq \tau = 0$ let us write $\Phi(t) = \Phi(t, 0)$. By time-invariance of (A, N, B, C), for arbitrary $K \geq 0$ we have

$$\mathcal{E}(\Phi(t, \tau)) = \mathcal{E}(\Phi(t - \tau)) \ ,$$

$$\mathcal{E}(\Phi(t, \tau)K\Phi(t, \tau)^T) = \mathcal{E}(\Phi(t - \tau)K\Phi(t - \tau)^T) \ .$$

Let $\mathbb{L} : L^2_w(\mathbb{R}_+, \mathbb{R}^m) \to L^2_w(\mathbb{R}_+, \mathbb{R}^p)$ denote the input-output operator, which maps input signals u to output signals y if $x(0) = 0$ (see [6]). By the stability assumption the Lyapunov equations (4) and (5) have nonnegative definite solutions $P, Q \geq 0$, which can be written (cf. [4]) as

$$P = \mathcal{E}\left(\int_0^\infty \Phi(t)BB^T\Phi(t)^T\,dt\right) \tag{7}$$

$$Q = \mathcal{E}\left(\int_0^\infty \Phi(t)^TC^TC\Phi(t)\,dt\right) \ . \tag{8}$$

Let $x_0 \in \mathbb{R}^n$ be given and write $x(t) = x(t, x_0, u)$ for the solution with initial value $x(0) = x_0$ and input u. We determine the minimal energy of an input u, so that $\mathcal{E}(x(T, 0, u)) = x_0$ for some $T > 0$, that is, u steers the mean state from 0 to x_0 over an arbitrary time interval $[0, T]$. Similarly, we consider the output energy produced by x_0 and define the energy functionals

$$E_c(x_0) = \inf_{\substack{u\in L^2_w[0,T], T>0 \\ \mathcal{E}(x(T,x_0,u))=0}} \mathcal{E}\left(\int_0^T \|u(t)\|^2\,dt\right) \ ,$$

$$E_o(x_0) = \mathcal{E}\left(\int_0^\infty \|y(t, x_0, 0)\|^2\,dt\right) \ .$$

Note that $E_c(x_0) = \infty$, if the mean state x_0 cannot be reached from 0. It is easy to see that this is equivalent to $x_0 \notin \operatorname{Im} P$. We have the following characterization of the energies (where $P^\dagger$ denotes the Moore-Penrose-Inverse).

Theorem II.1 *Consider the stochastic system* (6) *and the Gramians P and Q defined by* (4), (5). *If $x_0 \in \operatorname{Im} P$ then*

$$E_c(x_0) = x_0^T P^\dagger x_0 \ .$$

For arbitrary $x_0 \in \mathbb{R}^n$ we have $E_o(x_0) = x_0^T Q x_0$.

Proof: For $T > 0$ let $P_T = \mathcal{E}\left(\int_0^T \Phi(t)BB^T\Phi(t)^T\,dt\right)$ and for fixed $x_0 \in \operatorname{Im} P_T = \operatorname{Im} P$, define $u : [0,T] \to \mathbb{R}^m$ via $u(t) = B^T\Phi(T,t)^T P_T^\dagger x_0$. Then we have

$$\begin{aligned}
\mathcal{E}\left(x(T,0,u)\right) &= \mathcal{E}\Big(\int_0^T \Phi(T,t)Bu(t)\,dt\Big)\\
&= \mathcal{E}\Big(\int_0^T \Phi(T,t)BB^T\Phi(T,t)^T P_T^\dagger x_0\,dt\Big)\\
&= \mathcal{E}\Big(\int_0^T \Phi(T-t)BB^T\Phi(T-t)^T P_T^\dagger x_0\,dt\Big)\\
&= \mathcal{E}\Big(\int_0^T \Phi(\tau)BB^T\Phi(\tau)^T\,d\tau\Big)P_T^\dagger x_0\\
&= P_T P_T^\dagger x_0 = x_0 \ .
\end{aligned}$$

Moreover, u is the unique control with $\mathcal{E}(x(T,0,u)) = x_0$ and minimal $L_w^2[0,T]$-norm

$$\begin{aligned}
\|u\|_{L_w^2}^2 &= \mathcal{E}\Big(\int_0^T \|u(t)\|^2\,dt\Big)\\
&= \mathcal{E}\Big(\int_0^T x_0^T P_T^\dagger \Phi(t)BB^T\Phi(t)^T P_T^\dagger x_0\,dt\Big)\\
&= x_0^T P_T^\dagger P_T P_T^\dagger x_0 = x_0^T P_T^\dagger x_0 \ .
\end{aligned}$$

To prove minimality, assume that $\tilde{u} = u + \hat{u}$ is another solution to the control problem. Then

$$\begin{aligned}
x_0 &= \mathcal{E}\Big(\int_0^T \Phi(T,t)B(u(t)+\hat{u}(t))\,dt\Big) \ ,\\
\Rightarrow \qquad & \mathcal{E}\Big(\int_0^T \Phi(T,t)B\hat{u}(t)\,dt\Big) = 0 \ .
\end{aligned}$$

This implies $\mathcal{E}\Big(\int_0^T u(t)^T\hat{u}(t)\,dt\Big) = 0$, so that

$$\|\tilde{u}\|_{L_w^2[0,T]}^2 = \|u\|_{L_w^2[0,T]}^2 + \|\hat{u}\|_{L_w^2[0,T]}^2 \geq \|u\|_{L_w^2[0,T]}^2 \ .$$

Hence $E_c(x_0) = \inf_{T>0} x_0^T P_T^\dagger x_0$. By the definitions, P_T is monotonically increasing and $\lim_{T\to\infty} P_T = P$. Hence $P_T^\dagger$ is decreasing and the infimum is given by $x_0^T P^\dagger x_0$.
On the other hand, if the system starts in state x_0 and is not controlled, then the corresponding output is $y(t) = C\Phi(t)x_0$. Thus

$$E_o(x_0) = x_0^T\mathcal{E}\Big(\int_0^\infty \Phi(t)^T C^T C\Phi(t)\,dt\Big)x_0 = x_0^T Q x_0$$

which concludes the proof. □

Based on these Gramians we can apply balanced truncation to the given stochastic system. Under a similarity transformation

$$(A,N,B,C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

the Gramians transform contragrediently as

$$(P,Q) \mapsto (S^{-1}PS^{-T}, S^TQS) \ .$$

Choosing e.g. $S = LV\Sigma^{-1/2}$, with Cholesky factorizations $LL^T = P$, $R^TR = Q$ and a singular value decomposition $RL = U\Sigma V^T$, we have $S^{-1} = \Sigma^{-1/2}U^TR$ and

$$S^{-1}PS^{-T} = S^TQS = \Sigma = \operatorname{diag}(\sigma_1,\ldots,\sigma_n) \ .$$

The numbers $\sigma_1 \geq \ldots \geq \sigma_n \geq 0$ can be seen as generalized Hankel singular values. After suitable partitioning

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0\\ 0 & \Sigma_2 \end{bmatrix}, \ S = \begin{bmatrix} S_1 & S_2 \end{bmatrix}, \ S^{-1} = \begin{bmatrix} T_1\\ T_2 \end{bmatrix}$$

a truncated system is given in the form

$$(A_{11}, N_{11}, B_1, C_1) = (T_1AS_1, T_1NS_1, T_1B, CS_1) \ .$$

If the diagonal entries of Σ_2 are small, then those states have been cut off, which are both hard to reach and hard to observe. Hence it is expected that the truncation error is small. In fact this is supported by an H^2-error bound obtained in [12]. Additionally, however, from the deterministic situation (see [9], [1]), one would also hope for an H^∞-type error bound of the form

$$\|y - y_r\|_{L_w^2(\mathbb{R}_+,\mathbb{R}^p)} \overset{?}{\leq} \alpha \operatorname{trace}\Sigma_2 \|u\|_{L_w^2(\mathbb{R}_+,\mathbb{R}^m)} \qquad (9)$$

with some number $\alpha > 0$. Unfortunately, the following example shows that no such general α exists.

Example II.2 Let $A = \begin{bmatrix} -1 & 0\\ 0 & -a^2 \end{bmatrix}$, $N = \begin{bmatrix} 0 & 0\\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1\\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$, where $a > 1$.
The Gramians are $P = \begin{bmatrix} \frac{1}{2} & 0\\ 0 & \frac{1}{4a^2} \end{bmatrix}$, $Q = \begin{bmatrix} \frac{1}{4a^2} & 0\\ 0 & \frac{1}{2a^2} \end{bmatrix}$ with $\sigma(PQ) = \{\frac{1}{8a^2}, \frac{1}{8a^4}\}$ so that $\Sigma = \operatorname{diag}(\sigma_1,\sigma_2)$, where $\sigma_1 = \frac{1}{\sqrt{8}a}$ and $\sigma_2 = \frac{1}{\sqrt{8}a^2}$. The system is balanced by the transformation $S = \begin{bmatrix} 2a^2 & 0\\ 0 & 1/2 \end{bmatrix}^{1/4}$.
Then $CS = \frac{1}{2^{1/4}}\begin{bmatrix} 0 & 1 \end{bmatrix}$ so that $C_r = 0$ for the truncated system of order 1. Thus the output of the reduced system is $y_r \equiv 0$, and the truncation error $\|\mathbb{L} - \mathbb{L}_r\|$ is equal to the stochastic H^∞-norm (see [6]) of the original system,

$$\|\mathbb{L}\| = \sup_{x(0)=0,\|u\|_{L_w^2}=1} \|y\|_{L_w^2} \ .$$

We show now that this norm is equal to $\frac{1}{\sqrt{2}a} = 2a\sigma_2$. Thus, depending on a, the ratio of the truncation error and $\operatorname{trace}\Sigma_2 = \sigma_2$ can be arbitrarily large.

According to the stochastic bounded real lemma (see [6]), $\|\mathbb{L}\|$ is the infimum over all γ so that the Riccati inequality

$$0 < A^TX + XA + N^TXN - C^TC - \frac{1}{\gamma^2}XBB^TX \quad (10)$$
$$= \begin{bmatrix} -2x_1 + x_3 - \frac{1}{\gamma^2}x_1^2 & -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 \\ -(a^2+1)x_2 - \frac{1}{\gamma^2}x_1x_2 & -2a^2x_3 - \frac{1}{\gamma^2}x_2^2 - 1 \end{bmatrix}$$

possesses a solution $X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} < 0$.

If a given matrix X satisfies this condition then so does the same matrix with x_2 replaced by 0. Hence we can assume that $x_2 = 0$, and end up with the two conditions $x_3 < -\frac{1}{2a^2}$ and (after multiplying the upper left entry with $-\gamma^2$)

$$\begin{aligned} 0 &> x_1^2 + 2\gamma^2x_1 - \gamma^2x_3 = (x_1+\gamma^2)^2 - \gamma^2(\gamma^2 + x_3) \\ &> (x_1+\gamma^2)^2 - \gamma^2(\gamma^2 - \tfrac{1}{2a^2}) \ . \end{aligned}$$

Thus necessarily $\gamma^2 > \frac{1}{2a^2}$, i.e. $\gamma > \frac{1}{\sqrt{2}a}$. This already proves that $\|\mathbb{L}\| \geq \frac{1}{\sqrt{2}a} = 2a\sigma_2$, which suffices to disprove the existence of a general bound α in (9). Taking infima, it is easy to show that indeed $\|\mathbb{L}\| = \frac{1}{\sqrt{2}a}$.

We will now introduce a modification of the Gramians, for which we can prove a bound of the form (9) with $\alpha = 2$.

III. A NEW CONTROLLABILITY GRAMIAN AND A GUARANTEED ERROR BOUND

We have seen that the control energy actually is characterized by P^{-1} (or $P^\dagger$) rather than P. In the deterministic case, P^{-1} satisfies the Bernoulli equation

$$A^TP^{-1} + P^{-1}A = -P^{-1}BB^TP^{-1} \ . \quad (11)$$

On the left side of the equation, we have the Lyapunov operator $X \mapsto A^TX + XA$ of the observability Gramian applied to P^{-1}, while on the right we have $-P^{-1}BB^TP^{-1}$.

We transfer this formal recipe to the asymptotically mean-square stable stochastic linear system (6).

The Lyapunov operator of the observability Gramian now takes the form $X \mapsto A^TX + XA + N^TXN$. Thus, a formal analogue of equation (11) (different from (4)) is given by

$$A^T\hat{P}^{-1} + \hat{P}^{-1}A + N^T\hat{P}^{-1}N = -\hat{P}^{-1}BB^T\hat{P}^{-1} \ , \quad (12)$$

or equivalently (if $\hat{P} > 0$)

$$\hat{P}A^T + A\hat{P} + \hat{P}N^T\hat{P}^{-1}N\hat{P} = -BB^T \ . \quad (13)$$

Criteria for the existence of a positive definite matrix $\hat{P}$ satisfying this equation still have to be clarified. But actually (see (17) below), we will consider the *inequality*

$$A^T\hat{P}^{-1} + \hat{P}^{-1}A + N^T\hat{P}^{-1}N + \hat{P}^{-1}BB^T\hat{P}^{-1} \leq 0, \quad (14)$$

which is always solvable under our stability assumption. For brevity, we call $\hat{P} > 0$ satisfying (14) a *(new) Gramian*, too.

Lemma III.1 *Assume that $dx = Ax\,dt + Nx\,dw$ is asymptotically mean-square-stable.*
Then inequality (14) *is solvable with $\hat{P} > 0$.*

Proof: By the stability assumption, for a given $Y < 0$, there exists a $\tilde{P} > 0$, so that $A^T\tilde{P}^{-1} + \tilde{P}^{-1}A + N^T\tilde{P}^{-1}N = Y$ (see (7), or e.g. [4, Thm. 3.6.1], [13]). Then $\hat{P} = \varepsilon^{-1}\tilde{P}$, for sufficiently small $\varepsilon > 0$, satisfies

$$A^T\hat{P}^{-1} + \hat{P}^{-1}A + N^T\hat{P}^{-1}N = \varepsilon Y < -\varepsilon^2\tilde{P}^{-1}BB^T\tilde{P}^{-1}$$

so that (14) holds even in the strict form. □

Remark III.2 (a) *In view of our application, we aim at a Gramian $\hat{P}$, so that (some of) the eigenvalues of $\hat{P}Q$ are particularly small, since they provide the error bound. Choosing a very small ε in the previous proof can be contrary to this aim. Hence some optimization over all solutions of* (14) *can be required.*

(b) *Note also that a matrix $\hat{P} > 0$ satisfies* (14)*, if and only if it satisfies the linear matrix inequality*

$$\begin{bmatrix} \hat{P}A^T + A\hat{P} + BB^T & \hat{P}N^T \\ N\hat{P} & -\hat{P} \end{bmatrix} \leq 0 \ . \quad (15)$$

Thus, LMI optimal solution techniques are applicable to solve the problem raised in (a) (see Example IV-B). However, we need to gain control of the numerical complexity. This can be a topic for further investigations.

It is easy to see that a state space transformation

$$(A, N, B, C) \mapsto (S^{-1}AS, S^{-1}NS, S^{-1}B, CS)$$

leads to a contragredient transformation of the Gramians, $Q \mapsto S^TQS$, $\hat{P} \mapsto S^{-1}\hat{P}S^{-T}$. That is, $\hat{P}$ satisfies (14), if and only if $S^{-1}\hat{P}S^{-T}$ does so for the transformed data. As before, we can assume the system to be balanced with

$$\hat{P} = Q = \Sigma = \mathrm{diag}(\sigma_1 I, \ldots, \sigma_\nu I) = \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \ , \quad (16)$$

where $\sigma_1 > \sigma_2 > \ldots > \sigma_\nu > 0$ and $\sigma(\Sigma_1) = \{\sigma_1, \ldots, \sigma_r\}$, $\sigma(\Sigma_2) = \{\sigma_{r+1}, \ldots, \sigma_\nu\}$. Again, we call the numbers σ_j which are the positive square roots of the eigenvalues of $\hat{P}Q$ (generalized) Hankel singular values of the system. Partitioning A, N, B, C like Σ, we write the system as

$$\begin{aligned} dx_1 &= (A_{11}x_1 + A_{12}x_2)\,dt + (N_{11}x_1 + N_{12}x_2)\,dw + B_1u \\ dx_2 &= (A_{21}x_1 + A_{22}x_2)\,dt + (N_{21}x_1 + N_{22}x_2)\,dw + B_2u \\ y &= C_1x_1 + C_2x_2 \ . \end{aligned}$$

The reduced system obtained by truncation is

$$dx_r = A_{11}x_r + N_{11}x_r\,dw + B_1u \ , \quad y_r = C_1x_r \ .$$

The index r is the number of different singular values that have been kept in the reduced system.
Note that $\hat{P}_1 = Q_1 = \Sigma_1$ only satisfy the *inequalities*

$$\begin{aligned} A_{11}^T\Sigma_1^{-1} + \Sigma_1^{-1}A_{11} + N_{11}^T\Sigma_1^{-1}N_{11} &\leq -\Sigma_1^{-1}B_1B_1^T\Sigma_1^{-1} \\ A_{11}^T\Sigma_1 + \Sigma_1A_{11} + N_{11}^T\Sigma_1N_{11} &\leq -C_1^TC_1 \end{aligned}$$

even if $\hat{P}$ and Q satisfy the *equations* (12) and (5).

Hence, we will now assume (after balancing) that a diagonal matrix Σ as in (16) is given which satisfies

$$\begin{aligned} A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N &\leq -\Sigma^{-1}BB^T\Sigma^{-1} \\ A^T\Sigma + \Sigma A + N^T\Sigma N &\leq -C^TC \ . \end{aligned} \tag{17}$$

All partitionings and truncations are defined as above with respect to this Σ.

Theorem III.3 *If $x(0) = 0$ and $x_r(0) = 0$, then for all $T > 0$, we have*

$$\|y - y_r\|_{L^2_w[0,T]} \leq 2(\sigma_{r+1} + \ldots + \sigma_\nu)\|u\|_{L^2_w[0,T]} \ .$$

Proof: Recall a general consequence of Itô's Lemma: If $d\xi = A\xi\, dt + N\xi\, dw + Bu$ and $V \in \mathbb{R}^{n\times n}$, then

$$\begin{aligned} \frac{d}{dt}\mathcal{E}(\xi^TV\xi) = \mathcal{E}\Big(&\xi^T(A^TV + VA + N^TVN)\xi \\ &+ \xi^TA^TVBu + u^TB^TVA\xi\Big) \ . \end{aligned} \tag{18}$$

Thus (with $\xi = x_1 - x_r$ and $V = \Sigma_1$)

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}\left((x_1 - x_r)^T\Sigma_1(x_1 - x_r)\right) \\ &= 2\mathcal{E}\left((x_1 - x_r)^T\Sigma_1([A_{11}, A_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right])\right) \\ &+ \mathcal{E}\left(([N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_1([N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right])\right) \end{aligned}$$

and (with $\xi = x_2$ and $V = \Sigma_2$)

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}\left(x_2^T\Sigma_2x_2\right) = 2\mathcal{E}\left(x_2^T\Sigma_2(A_{21}x_1 + A_{22}x_2 + B_2u)\right) \\ &+ \ \mathcal{E}\left((N_{21}x_1 + N_{22}x_2)^T\Sigma_2(N_{21}x_1 + N_{22}x_2)\right) \ . \end{aligned}$$

Now consider $y - y_r = [C_1, C_2]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right]$. From the second inequality in (17) we obtain

$$\begin{aligned} &- \mathcal{E}\left(\|y - y_r\|^2\right) = -\mathcal{E}\left[\begin{array}{c} x_1 - x_r \\ x_2 \end{array}\right]^T C^TC\left[\begin{array}{c} x_1 - x_r \\ x_2 \end{array}\right] \\ &\geq \mathcal{E}\left[\begin{array}{c} x_1 - x_r \\ x_2 \end{array}\right]^T\Big(A^T\Sigma + \Sigma A + N^T\Sigma N\Big)\left[\begin{array}{c} x_1 - x_r \\ x_2 \end{array}\right] \\ &= 2\mathcal{E}\left((x_1 - x_r)^T\Sigma_1[A_{11}, A_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right]\right) \\ &+ \ 2\mathcal{E}\left(x_2^T\Sigma_2[A_{21}, A_{22}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right]\right) \\ &+ \ \mathcal{E}\left(([N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_1[N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right]\right) \\ &+ \ \mathcal{E}\left(([N_{21}, N_{22}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_2[N_{21}, N_{22}]\left[\begin{smallmatrix} x_1 - x_r \\ x_2 \end{smallmatrix}\right]\right) \ . \end{aligned}$$

Comparing these equations we find that

$$\begin{aligned} &- \mathcal{E}\left(\|y - y_r\|^2\right) \geq \frac{d}{dt}\mathcal{E}\left((x_1 - x_r)^T\Sigma_1(x_1 - x_r)\right) \\ &+ \frac{d}{dt}\mathcal{E}\left(x_2^T\Sigma_2x_2\right) - \ 2\mathcal{E}\left(x_2^T\Sigma_2(A_{21}x_r + B_2u)\right) \\ &- \ \mathcal{E}\left((N_{21}x_r)^T\Sigma_2(2N_{21}x_1 + 2N_{22}x_2 - N_{21}x_r)\right) \ . \end{aligned}$$

Since $x(0) = 0$ and $x_r(0) = 0$ it follows that

$$\begin{aligned} &\int_0^T \mathcal{E}\left(\|y - y_r\|^2\right)\, dt \\ &\leq 2\int_0^T \mathcal{E}\left(x_2^T\Sigma_2(A_{21}x_r + B_2u)\right)\, dt \\ &+ \ 2\int_0^T \mathcal{E}\left((N_{21}x_r)^T\Sigma_2(N_{21}x_1 + N_{22}x_2)\right)\, dt \ . \end{aligned} \tag{19}$$

Our next step is to show that

$$\begin{aligned} &4\int_0^T \mathcal{E}\left(\|u\|^2\right)\, dt \geq 2\int_0^T \mathcal{E}\left(x_2^T\Sigma_2^{-1}(A_{21}x_r + B_2u)\right)\, dt \\ &+ \ 2\int_0^T \mathcal{E}\left((N_{21}x_r)^T\Sigma_2^{-1}(N_{21}x_1 + N_{22}x_2)\right)\, dt \ . \end{aligned} \tag{20}$$

To this end we consider the first inequality in (17). By the Schur complement definiteness criterion it implies that

$$\left[\begin{array}{cc} A^T\Sigma^{-1} + \Sigma^{-1}A + N^T\Sigma^{-1}N & \Sigma^{-1}B \\ B^T\Sigma^{-1} & -I \end{array}\right] \leq 0 \ .$$

This again is equivalent to

$$\begin{aligned} \left[\begin{array}{cc} 0 & 0 \\ 0 & I \end{array}\right] \geq \left[\begin{array}{cc} A & B \\ I & 0 \end{array}\right]^T\left[\begin{array}{cc} 0 & \Sigma^{-1} \\ \Sigma^{-1} & 0 \end{array}\right]\left[\begin{array}{cc} A & B \\ I & 0 \end{array}\right] \\ + \left[\begin{array}{cc} N^T\Sigma^{-1}N & 0 \\ 0 & 0 \end{array}\right] \ . \end{aligned}$$

We multiply this inequality from the right by $\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \\ 2u \end{smallmatrix}\right]$ and from the left by $\left[\begin{smallmatrix} x_1 + x_r & x_2 & 2u \end{smallmatrix}\right]$. Taking expectations, we get

$$\begin{aligned} &4\mathcal{E}\left(\|u\|^2\right) \\ &\geq 2\mathcal{E}\left((x_1 + x_r)^T\Sigma_1^{-1}([A_{11}, A_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right] + 2B_1u)\right) \\ &+ \ 2\mathcal{E}\left(x_2^T\Sigma_2^{-1}([A_{21}, A_{22}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right] + 2B_2u)\right) \\ &+ \ \mathcal{E}\left(([N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_1^{-1}[N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right]\right) \\ &+ \ \mathcal{E}\left([N_{21}, N_{22}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_2^{-1}[N_{21}, N_{22}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right]\right) \ . \end{aligned}$$

Like above (using (18)), we compare these terms with

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}\left((x_1 + x_r)^T\Sigma_1^{-1}(x_1 + x_r)\right) \\ &= 2\mathcal{E}\left((x_1 + x_r)^T\Sigma_1^{-1}([A_{11}, A_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right] + 2B_1u)\right) \\ &+ \ \mathcal{E}\left(([N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right])^T\Sigma_1^{-1}[N_{11}, N_{12}]\left[\begin{smallmatrix} x_1 + x_r \\ x_2 \end{smallmatrix}\right]\right) \ , \end{aligned}$$

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}\left(x_2^T\Sigma_2^{-1}x_2\right) = 2\mathcal{E}\left(x_2^T\Sigma_2^{-1}(A_{21}x_1 + A_{22}x_2 + B_2u)\right) \\ &+ \ \mathcal{E}\left((N_{21}x_1 + N_{22}x_2)^T\Sigma_2^{-1}(N_{21}x_1 + N_{22}x_2)\right) \ . \end{aligned}$$

Then (20) follows from

$$\begin{aligned} 4\mathcal{E}\left(\|u\|^2\right) &\geq \frac{d}{dt}\mathcal{E}\left((x_1 + x_r)^T\Sigma_1^{-1}(x_1 + x_r)\right) \\ &+ \ \frac{d}{dt}\mathcal{E}\left(x_2^T\Sigma_2^{-1}x_2\right) + 2\mathcal{E}\left(x_2^T\Sigma_2^{-1}(A_{21}x_r + B_2u)\right) \\ &+ \ \mathcal{E}\left((N_{21}x_r)^T\Sigma_2(2N_{21}x_1 + 2N_{22}x_2 + N_{21}x_r)\right) \ . \end{aligned}$$

If $\Sigma_2 = \sigma_\nu I$, then $\Sigma_2 = \sigma_\nu^2\Sigma_2^{-1}$. In this case, the right hand side of (20) multiplied with σ_ν^2 equals the right hand side of (19). Hence, for all $T > 0$ we have

$$\|y - y_{\nu-1}\|_{L^2_w[0,T]} \leq 2\sigma_\nu\|u\|_{L^2_w[0,T]} \ .$$

We can repeat this procedure step by step to remove $\sigma_{\nu-1}, \ldots, \sigma_{r+1}$. By the triangle inequality we find that

$$\begin{aligned} \|y - y_r\|_{L^2_w[0,T]} &\leq \sum_{j=r}^{\nu-1}\|y_{j+1} - y_j\|_{L^2_w[0,T]} \\ &\leq 2(\sigma_\nu + \ldots + \sigma_{r+1})\|u\|_{L^2_w[0,T]} \ . \end{aligned}$$

which concludes the proof. □

Example III.4 Let the system (A, N, B, C) and the Gramian Q be as in Example II.2. The matrix

$$\hat{P} = \begin{bmatrix} 1+\sqrt{1-\hat{p}} & 0 \\ 0 & \hat{p} \end{bmatrix}^{-1} > 0\ , \text{ where } 0 < \hat{p} \leq 1\ ,$$

satisfies inequality (14). As in Example II.2, we have $\mathbb{L}_r = 0$ for the corresponding reduced system of order 1, so that the truncation error again is $\frac{1}{\sqrt{2}a}$, independently of $\hat{p} \in]0, 1]$.

On the other hand we have

$$\sigma_2^2 = \min \sigma(\hat{P}Q) = \frac{1}{4a^2(1+\sqrt{1-\hat{p}})} \leq \frac{1}{8a^2}\ ,$$

with equality for $\hat{p} \to 0$. Theorem III.3 thus gives the sharp error bound $2\sigma_2 = \frac{1}{\sqrt{2}a}$. Note, that there is no $\hat{P} > 0$ satisfying the *equation* (13).

The previous example illustrates the problem of optimizing over all solutions of inequality (14).

IV. NUMERICAL EXAMPLES

As we have just seen, the new controllability Gramian satisfying equation (13) may not exist. Therefore, in general, it is better to work with the inequality (14). On the other hand, it is instructive to consider systems where both types of controllability Gramians exist. In our first example we construct such systems artificially. In the second example we consider a discretized two-dimensional heat equation with stochastic boundary effects and compute a solution of inequality (14) with an LMI-solver.

By $\mathbb{L}$ and $\mathbb{L}_r$, we always denote the original and the r-th order approximated system. The stochastic H^∞-type norm $\|\mathbb{L} - \mathbb{L}_r\|$ is computed by a binary search of the infimum of all γ such that the Riccati inequality (10) is solvable. The latter is solved via a Newton iteration as in [4]. Finally, the Lyapunov equations (2) are solved by preconditioned Krylov subspace methods described in [5].

A. Systems with known Gramians

Systems, for which we know in advance that both equations

$$AP + PA^T + NPN^T = -BB^T \qquad (21)$$

$$A\hat{P} + \hat{P}A^T + \hat{P}N\hat{P}^{-1}N^T\hat{P} = -BB^T \qquad (22)$$

possess solutions $\hat{P}, P > 0$, can be designed as follows.

We start with a mean-square stable stochastic system (A, N) and some generic matrix $\tilde{B}$ (satisfying a controllability condition). Then we solve

$$ZA + A^TZ + N^TZN + \tilde{B}\tilde{B}^T = 0\ , \qquad (23)$$

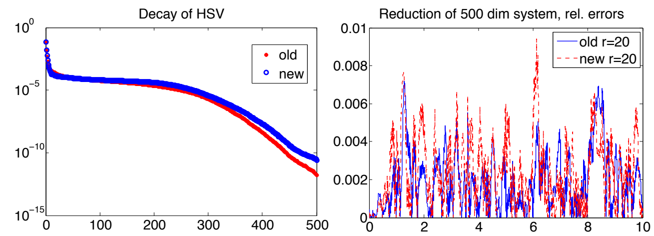
for $Z > 0$, which is possible by the stochastic version of Lyapunov's matrix theorem. Setting $\hat{P} = Z^{-1} > 0$, $B = \hat{P}\tilde{B}$, and multiplying (23) from both sides with $\hat{P}$, we have

$$A\hat{P} + \hat{P}A^T + \hat{P}N^T\hat{P}^{-1}N\hat{P} + BB^T = 0\ ,$$

i.e. $\hat{P}$ is the new controllability Gramian of the system with the modified B-matrix, given by (A, N, B). For this system we can also compute the old Gramians P and Q according to (4), (5). Clearly, this construction of B is artificial, but it provides us with some examples to compare.

We chose random data $A, N \in \mathbb{R}^{500\times 500}$ with $\sigma(A) \subset \mathbb{C}_-$ and N scaled such that the stability assumption is satisfied, $B \in \mathbb{R}^{500\times 50}$ and C the vector of all ones in $\mathbb{R}^{1\times 500}$.

In the following two figures we compare the reduced systems in both cases, where *old* and *new* refer to the old and new Gramian definitions in (4) and (12), respectively. The left figure shows the decay of the singular values. The right figure shows the relative difference $\frac{\|y(t)-y_r(t)\|}{\|y(t)\|}$ between the original output and the outputs of the reduced system over a given time interval. In fact, for many examples we have observed both methods to yield very similar results.



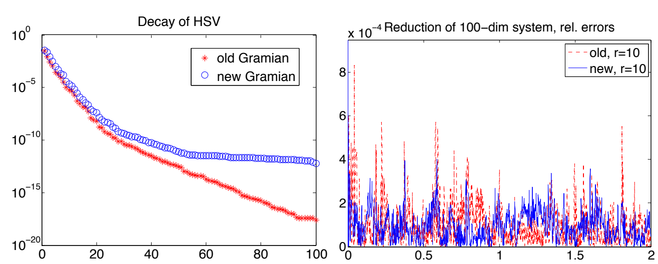
The estimated error norm and the actual approximation error are given in the following table for both cases:

	$\sum_{j=r+1}^{n} \sigma_j$	$\|\mathbb{L} - \mathbb{L}_r\|$
old	0.0243	0.0061
new	0.0285	0.0018

B. A heat transfer problem

We consider a stochastic modification of the heat transfer problem described in [3]. On the unit square $\Omega = [0, 1]^2$ the heat equation $x_t = \Delta x$ is given with Dirichlet condition $x = u_j$, $j = 1, 2, 3$ on three of the boundary edges and a stochastic Robin condition $n\cdot\nabla x = (1/2+\dot{w})x$ on the fourth edge (where $\dot{w}$ stands for white noise). A standard 5-point finite difference discretization on a 10×10 grid leads to a modified Poisson matrix $A \in \mathbb{R}^{100\times 100}$ and corresponding matrices $N \in \mathbb{R}^{100\times 100}$ and $B \in \mathbb{R}^{100\times 3}$. We use the input $u \equiv \begin{bmatrix}1\\1\\1\end{bmatrix}$ and choose the average temperature as the output, i.e. $C = \frac{1}{100}[1, \ldots, 1]$. As before, we apply balanced truncation based on the old Gramians P and Q and on new Gramians $\hat{P}$ and Q. But now we use an LMI-solver (MATLAB® function `mincx`) to compute $\hat{P}$ as a solution of the linear matrix inequality (15) which minimizes trace $\hat{P}Q$.

The following figures show the analog curves as above.



Again, we have computed the estimated error norm and the actual approximation error for both cases:

	$\sum_{j=r+1}^{n}\sigma_j$	$\|\mathbb{L}-\mathbb{L}_r\|$
old	$1.47e-05$	$1.95e-05$
new	$5.35e-05$	$1.34e-05$

As we can see, the upper error bound fails for the old Gramian. For the new Gramian, it is correct and, in fact, the new method yields a slightly better approximation than the old method.

Clearly, higher dimensional examples are required to get more insight. To this end a more sophisticated method for the solution of (15) is needed. With general purpose LMI-software on a standard Laptop, we hardly got higher than $n = 100$.

V. CONCLUSIONS

We have introduced a new type of Gramians for stochastic linear systems and showed that balanced truncation based on these Gramians leads to an error bound which is analogous to the one known from deterministic systems. Further analysis and computational issues are topics of future research.

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