Towards Riccati-Feedback Control of Complex Flows with Moving Interfaces

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Problems featuring moving interfaces appear in many applications. They can model solidification and melting of pure materials, crystal growth and other multi-phase problems. The control of the moving interface enables to, for example, influence production processes and, thus, the product material quality.

We consider the two-phase Stefan problem that models a solid and a liquid phase separated by the moving interface. In the liquid phase, the heat distribution is characterized by a convection-diffusion equation. The fluid flow in the liquid phase is described by the Navier–Stokes equations which introduces a differential algebraic structure to the system. The interface movement is coupled with the temperature through the Stefan condition, which adds additional algebraic constraints. Our formulation uses a sharp interface representation and we define a quadratic tracking-type cost functional as a target of a control input.

We compute an open loop optimal control for the Stefan problem using an adjoint system. For a feedback representation, we linearize the system about the trajectory defined by the open loop control. This results in a linear-quadratic regulator problem, for which we formulate the differential Riccati equation with time varying coefficients. This Riccati equation defines the corresponding feedback gain.

Further, we present the feedback formulation that takes into account the structure and the differential algebraic components of the problem. Also, we discuss how the complexities that come, for example, with mesh movements, can be handled in a feedback setting.

1 Feedback Control of a Stefan Problem

We aim to control and stabilize a coupled nonlinear system. As a exemplary coupled nonlinear system we use a two-phase two-dimensional Stefan problem. We compute a reference solution \((\tilde{x}, \tilde{u})\) with an open loop control as from [1]. Since the open loop control \(\tilde{u}\) is not robust against perturbations and uncertainties, it should be stabilized. We choose a Riccati feedback strategy to achieve the stabilization. The system is linearized around \((\tilde{x}, \tilde{u})\) which leads to the following linear-quadratic control problem:

\[
\begin{align*}
\text{Minimize} & \quad \mathcal{J}(y, u) = \frac{1}{2} \int_0^\infty ||y(t) - y_a(t)||^2 + \lambda ||u(t)||^2 \, dt \\
\text{subject to} & \quad \mathcal{M}(t) \frac{d}{dt} x(t) = A(t)x(t) + B(t)u(t), \\
& \quad y(t) = C(t)x(t). 
\end{align*}
\]

Here, \(x\) is the state, \(u\) the input, and \(y\) the output of the system. Details regarding the linearization and the matrices \(\mathcal{M}, A, B,\) and \(C\) are given in Section 2. We use mesh movement techniques to represent the interface and resolve the discontinuity of the temperature gradient along the interface. The spatial discretization mesh is updated in every time step with the mesh velocity \(V_{\text{mesh}}\) from equations \((2b)\) and \((2c)\). Thus, not only the functions in equation \((1)\) are time dependent but also the matrices. We omit the time dependence "\((t)\)" for the sake of brevity in what follows. The optimal control \(u = -Kx\) can be computed with the feedback matrix \(K = B^T \mathcal{X} \mathcal{M}\) [2]. The matrix \(\mathcal{X}\) is the solution of the generalized differential Riccati equation with time dependent coefficients.

\[
-\mathcal{M}^T \dot{\mathcal{X}} \mathcal{M} = C^T C + (\dot{A} + A)^T \mathcal{X} \mathcal{M} + \mathcal{M}^T \mathcal{X} (\dot{A} + A) - \mathcal{M}^T \mathcal{X} BB^T \mathcal{X} \mathcal{M}.
\]
2 Linearization of a Stefan Problem

For a linearization of a Stefan problem, we use known reference trajectories \((\bar{T}, \bar{V}_{\text{mesh}}, \bar{v})\) from open loop optimal control [1].

\[
\mathcal{M} d\frac{d}{dt} T + A_3(\bar{T}) v + A_5(\bar{T}) V_{\text{mesh}} + A_4 T, \quad \leftarrow \quad \frac{d}{dt} T + (v - V_{\text{mesh}}) \cdot \nabla \bar{T} - \alpha \Delta T = 0, \quad \text{on } \Omega, \quad \text{(2a)}
\]

\[
A_{7,1} V_{\text{mesh}}, \quad \leftarrow \quad \Delta V_{\text{mesh}} = 0, \quad \text{on } \Omega, \quad \text{(2b)}
\]

\[
A_{7,2} V_{\text{mesh}} + A_6 T, \quad \leftarrow \quad V_{\text{mesh}} - \left( \frac{1}{\mathcal{L}} \left[ k_s(\nabla T)_{\text{s}} - k_l(\nabla T)_{\text{l}} \right] \right) \cdot n_{\text{int}} = 0, \quad \text{on } \Gamma_{\text{int}}, \quad \text{(2c)}
\]

\[
\mathcal{M}_v \frac{d}{dt} v + (A_1(\bar{v}, \bar{V}_{\text{mesh}}) + A_{2,1}) v + J_{1,1} p, \quad \leftarrow \quad \frac{d}{dt} v + ((\bar{v} - \bar{V}_{\text{mesh}}) \cdot \nabla) v - \eta \Delta v + \nabla p = 0, \quad \text{on } \Omega_i, \quad \text{(2d)}
\]

\[
J_{2} v, \quad \leftarrow \quad \nabla \cdot v = 0, \quad \text{on } \Omega_i, \quad \text{(2e)}
\]

\[
J_{1,2} p + A_{2,2} v + B u, \quad \leftarrow \quad p \cdot n - \eta \theta_{\text{u}} v = \mathbf{u} \cdot \mathbf{n}, \quad \text{on } \Gamma_{\text{in}}, \quad \text{(2f)}
\]

\[
\mathcal{C}_f T, \quad \leftarrow \quad \frac{1}{\mathcal{L}} \left[ k_s(\nabla T)_{\text{s}} - k_l(\nabla T)_{\text{l}} \right] = \mathbf{y}, \quad \text{on } \Gamma_{\text{in}} \quad \text{(2g)}
\]

For the Stefan problem, the state \(x = [v, T, V_{\text{mesh}}, p]\) consists of the velocity of the fluid, the temperature, the mesh velocity, and the pressure respectively. The domain \(\Omega\) is split into the solid phase \(\Omega_{\text{s}}\) and the liquid phase \(\Omega_{\text{l}}\). The two phases are separated by the interface \(\Gamma_{\text{int}}\). The control \(u\) acts on the inflow boundary \(\Gamma_{\text{in}}\). The full system with all boundary conditions is given in [3]. The matrices in the left column of equation (2) result from the spatial discretization, with finite elements, of the weak formulations of the equations in (2). The weak formulations are defined similarly to [3]. The linear system in equation (1) has differential algebraic character resulting from the Navier–Stokes equations (2d) to (2f) and the Stefan condition (2c). The block matrix formulation reads

\[
\begin{bmatrix}
\mathcal{M}_v & 0 & 0 & 0 \\
0 & \mathcal{M}_T & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
T \\
V_{\text{mesh}} \\
p
\end{bmatrix}
+ \mathcal{A}
\begin{bmatrix}
A_1(\bar{v}, \bar{V}_{\text{mesh}}) + A_2 & 0 & 0 & J_1^T \\
0 & A_3(\bar{T}) & A_4 & A_5(\bar{T}) \\
0 & 0 & A_6 & 0 \\
J_2 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
T \\
V_{\text{mesh}} \\
p
\end{bmatrix}
+ B u,
\]

\[
y = \begin{bmatrix}
0 & \mathcal{C}_f & 0 & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
v \\
T \\
V_{\text{mesh}} \\
p
\end{bmatrix}.
\]

3 Conclusion

The Stefan problem can be solved numerically with mesh movement and finite elements. With this and a gradient algorithm, we can compute an open loop control and get a reference trajectory. In order to have a more robust control, we aim to use Riccati feedback stabilization for the Stefan problem. The linear quadratic regulator approach has been used successfully for Navier–Stokes equations and Diffusion-Convection Models, which are both part of the Stefan problem. For this approach, we linearize the nonlinear differential algebraic equations around the reference solution. This leads to a linear differential algebraic system with time dependent coefficients. Thus, the generalized differential Riccati equation has to be solved instead of the algebraic Riccati equation. Possible methods for this are, e.g., the Rosenbrock method and backward differentiation formula. For these, Lyapunov equations and algebraic Riccati equations need to be solved in every time step. Already existing methods for this are the Alternating Directions Implicit (ADI) iteration and the Newton-ADI. These solvers need to be adjusted to the structure and difficulties arising from the Stefan problem.

References