Damping Optimization for Linear Vibrating Systems Using Dimension Reduction

Peter Benner, Zoran Tomljanović, and Ninoslav Truhar

Abstract We consider a mathematical model of a linear vibrational system described by the second-order system of differential equations $M\ddot{x} + D\dot{x} + Kx = 0$, where M, K and D are positive definite matrices, called mass, stiffness and damping, respectively. We are interested in finding an optimal damping matrix which will damp a certain part of the undamped eigenfrequencies. For this we use a minimization criterion which minimizes the average total energy of the system. This is equivalent to the minimization of the trace of the solution of a corresponding Lyapunov equation. In this paper we consider an algorithm for the efficient optimization of the damping positions based on dimension reduction techniques. Numerical results illustrate the efficiency of our approach.
Keywords: vibrating system, Lyapunov equation, energy minimization, dimension reduction.

1 Introduction

The aim of this paper is the determination of optimal damping for the following linear vibrational system:

$$M\ddot{x} + D\dot{x} + Kx = 0, \tag{1}$$

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where M and K (called mass and stiffness, respectively) are real, symmetric positive definite matrices of order n. The damping matrix is defined as $D = C_u + C_{ext}$, where C_{ext} is the external damping. The internal damping C_u is usually taken to be a small multiple of the critical damping, more precisely [1]

$$C_u = \alpha_c C_{crit}, \text{ the critical damping is } C_{crit} = 2M^{1/2}\sqrt{M^{-1/2}KM^{-1/2}}M^{1/2}. \quad (2)$$

For the sake of simplicity, we will use parameter $\alpha = 2\alpha_c$. Equation (1) can be transformed to phase space which yields a system of first order differential equations. For this purpose, let Φ be a matrix that simultaneously diagonalizes M and K, that is

$$\Phi^T K\Phi = \Omega^2 = \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad \text{and} \quad \Phi^T M\Phi = I. \quad (3)$$

For the internal damping defined in (2) it holds that $\Phi^T C_u \Phi = \alpha\Omega$. The positive numbers $\omega_1, \omega_2, \ldots, \omega_n$ are the eigenvalues of the undamped system, also called undamped eigenfrequencies. Then, we can write the differential equation (1) in phase space as

$$\frac{d}{dt}y = Ay, \quad \text{where} \quad A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\Phi^T D\Phi \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad (4)$$

for more details see [1, 2, 3, 4]. The first order differential equation given above has the solution $y = e^{At}y_0$, where y_0 contains the initial data. It has been shown in [3, 5] that A from (4) is a stable matrix, that is, the eigenvalues of A are in the open left half of the complex plane.

The main aim is to determine the "best" damping matrix D which will insure optimal evanescence of each component of y. For this purpose, we will use the criterion of minimization of the total energy of the system, that is

$$\int_0^\infty E(t)dt \quad \to \quad \min, \quad (5)$$

where $E(t)$ is the total energy of the system at a given time t, as a sum of kinetic and potential energies. In [3] it is shown that by taking the average over all initial states of the unit total energy and a given frequency range, the minimization criterion (5) is equivalent to

$$\mathrm{trace}\, X \quad \to \quad \min, \quad (6)$$

where X is the solution of the Lyapunov equation $AX + XA^T = -GG^T$, with A as in (4). The matrix G depends on the eigenfrequencies which have to be damped. If we are interested in damping of all undamped eigenfrequencies, then $G = I$, while in the case of damping of just the first s eigenfrequencies of the undamped system, the matrix G has the following form [3]:

$$G = \begin{bmatrix} I_s & 0 & 0 & 0 \\ 0 & 0 & I_s & 0 \end{bmatrix}^T. \quad (7)$$

This optimization problem has been intensively considered, see for example [1, 2, 3, 4, 6, 7, 8, 9]. Optimal damping aims at optimization of the damping positions as well as the corresponding viscosities. In this paper we will be mainly interested in the optimal dampers' positions. The optimal viscosities can then be determined by an additional optimization procedure.

Let the external damping be given by

$$C_{ext} = v_1 e_{i_1} e_{i_1}^T + v_2 e_{i_2} e_{i_2}^T + \cdots + v_k e_{i_k} e_{i_k}^T, \tag{8}$$

where i_j, $j = 1,\ldots,k$, corresponds to the damping positions with viscosities v_j, $j = 1,\ldots,k$. It follows directly from Equation (8) that it is sufficient to find the optimal positions such that $1 \le i_1 < i_2 < \ldots < i_k \le n$. Since we are interested in determination of the optimal damping positions and viscosities, we will use a new notation for trace$X$ which is now a function of the damping positions $(i_1,\ldots,i_k)$ and the corresponding viscosities $(v_1,\ldots,v_k)$. Thus, let $X(C(v_1,\ldots,v_k;i_1,\ldots,i_k))$ be the solution of the Lyapunov equation

$$AX(C(v_1,\ldots,v_k;i_1,\ldots,i_k)) + X(C(v_1,\ldots,v_k;i_1,\ldots,i_k))A^T = -GG^T, \tag{9}$$

where $(i_1,\ldots,i_k)$ are the damping positions and $(v_1,\ldots,v_k)$ the corresponding viscosities. The matrix G is given in Equation (7), while the matrix A equals

$$A = \begin{bmatrix} 0 & \Omega \\ -\Omega & -\alpha\Phi^T C_{crit}\Phi - C(v_1,\ldots,v_k;i_1,\ldots,i_k) \end{bmatrix}, \tag{10}$$

where $C(v_1,\ldots,v_k;i_1,\ldots,i_k) = \Phi^T C_{ext}\Phi$ and Φ is the matrix given in Equation (3).

For a given mass matrix M, stiffness matrix K, internal damping C_u and k dampers, we are interested in determining the optimal positions $(i_1^{opt},\ldots,i_k^{opt})$ and corresponding viscosities $(v_1^{opt},\ldots,v_k^{opt})$ such that trace$X(C(v_1,\ldots,v_k;i_1,\ldots,i_k))$ is minimal.

In the next section we will discuss the main difficulties in the process of damping optimization and we suggest a new approach for efficient damping optimization.

2 Damping optimization by "discrete to continuous" optimization approach

This section will be mainly devoted to the calculation of the optimal damping positions. The problem of determining the optimal damping is extremely demanding, because numerous Lyapunov equations have to be solved. Furthermore, for systems with large dimensions, even solving a single Lyapunov equations with direct solvers (such as the Bartels-Stewart algorithm [10]) can become very demanding.

One approach for determination of the optimal damping positions is the "direct" approach, which includes viscosity optimization for all possible damping configurations. For the external damping given by Equation (8), we need to opti-

mize the viscosity at all different configurations of dampering positions such that $1 \le i_1 \le i_2 \le \ldots \le i_k \le n$. Then the optimal positions are those corresponding to the minimal trace$X$.

One heuristic optimization approach for the determination of the optimal damping positions is presented in [8]. In this paper, the authors group the possible damping positions in order to optimize the viscosities with respect to a smaller number of damping positions.

Here we will introduce the "discrete to continuous" heuristical approach which relies on the optimization of functions of real variables. First, we will define an additional function which will be used in the optimization procedure. We want to determine optimal damping for k dampers with different viscosities. Thus, for $\mathcal{D} \subset \mathbb{R}^{2k}$ we define a function $f : \mathcal{D} \to \mathbb{R}$ by

$$f(v_1,\ldots,v_k;i_1,\ldots,i_k) = \text{trace}(X(C(v_1,\ldots,v_k;[i_1],\ldots,[i_k]))), \tag{11}$$

where $[\cdot]$ stands for the rounding (we use the MATLAB® function `round`) and the matrix $X(C(v_1,\ldots,v_k;[i_1],\ldots,[i_k]))$ is the solution of the Lyapunov equation (9). Here i_k is considered is as a continuous variable and the damping positions $[i_1],[i_2],\ldots,[i_k]$ with corresponding viscosities $v_1,v_2,\ldots,v_k$ determine the matrix C.

Now, we reduce our optimization problem to the minimization of the function (11) with continuous domain.Thus, for minimization of this function we can use standard methods like Nelder-Mead method [11] (implemented in the MATLAB function `fminsearch`). One can also use a Newton-like methods or genetic algorithm, implemented in the MATLAB function `fmincon` or `ga`, respectively. In the optimization process we will use the Nelder-Mead method which is much more robust than the other mentioned methods for our minimization problem.

When we determine the minimum of the function (11) we will denote the point where the minimum is achieved by $(\hat{v}_1,\hat{v}_2,\ldots,\hat{v}_k;\hat{i}_1,\hat{i}_2,\ldots,\hat{i}_k)$. Then the optimal positions are $[\hat{i}_1],[\hat{i}_2],\ldots,[\hat{i}_k]$ with corresponding optimal viscosities equal to $\hat{v}_1,\hat{v}_2,\ldots,\hat{v}_k$.

A further question in minimization with a Nelder-Mead method is the choice of good starting points. First, we have to define the parameters d_3 and d_4 which determine the grid of starting points $(i_1^s,\ldots,i_k^s)$. As can be seen from Algorithm 1, the parameter d_3 determines the difference between points inside the region, while the parameter d_4 defines the distance to the edge of the region where the optimal position is to be found. Some fixed values $v_1^s,v_2^s,\ldots,v_k^s$ will be taken as starting viscosities.

Note that Nelder-Mead [11] is an unconstrained multidimensional optimization method. In numerical experiments the optimization procedure could require an evaluation at the points that are not in the domain (for example, viscosities may become negative). Thus, at points that are outside the domain where the optimization is performed, in our optimization procedure we set the function value to some constant large enough. With this, our optimization procedure will always return a minimum which is inside the domain of our value function.

Algorithm 1 ("Discrete to continuous" approach for determination of optimal positions)

Require: d_3, d_4 – parameters which determine the first and the second grid;
Ensure: Optimal dampers' positions $[i_1^{opt}], \ldots, [i_k^{opt}]$ with optimal viscosities $v_1^{opt}, \ldots, v_k^{opt}$.

1: **for** $i_1^s = 1 + d_4 : d_3 : n - d_4$ **do**
2: **for** $i_2^s = i_1^s + d_4 : d_3 : n - d_4$ **do**
3: $\cdots$
4: **for** $i_k^s = i_{k-1}^s + d_4 : d_3 : n - d_4$ **do**
5: Using starting points $(v_1^s, \ldots, v_k^s; i_1^s, \ldots, i_k^s)$ calculate (for example with Nelder-Mead algorithm)
$$\min_{\substack{(v_1,\ldots,v_k)\in\mathbb{R}_+^k \\ 1\le i_1<i_2<\ldots<i_k\le n}} f(v_1, \ldots, v_k; i_1, \ldots, i_k).$$
6: **end for**
7: $\cdots$
8: **end for**
9: **end for**
10: The parameters which correspond to the minimal value calculated in Step 5, are returned as optimal parameters $(v_1^{opt}, \ldots, v_k^{opt}; i_1^{opt}, \ldots, i_k^{opt})$.

3 Damping optimization based on dimension reduction and continuous minimization

We have introduced the "discrete to continuous" approach which can be applied to vibrating systems of moderate dimensions. However, for large systems solving of the corresponding Lyapunov equation is quite demanding itself. Thus, we propose a new approach which will combine approximation algorithms that use dimension reduction techniques with the "discrete to continuous" approach introduced in the previous section.

In case we intent to damp all undamped eigenfrequencies, we will use the approximation algorithms introduced in [6]. Contrary to this, in the case when we damp a selected part of the undamped eigenfrequencies, we will use the dimension reduction approach derived in [7].

In the minimization process with the "discrete to continuous" approach we cannot just directly apply the algorithms from [6, 7], since these algorithms optimize viscosities at given damping positions, while in the "discrete to continuous" approach we change the damping positions during the optimization process. Thus, we have to modify our algorithms which use a dimension reduction technique. This modification includes checking of the corresponding error bound at the each step of the optimization procedure. More precisely, for the approximation of the function given in Equation (11), we have to treat two cases which depend on the eigenfrequencies which have to be damped.

In the case of damping of selected eigenfrequencies, in Step 5 of Algorithm 1, we need to calculate an approximation of the function $f(v_1, \ldots, v_k; i_1, \ldots, i_k)$. Algorithm 2 gives an approximation with a given tolerance ε. In Algorithm 2, the parameter **u** represents the machine precision. For the purpose of simplification, on the input we give just parameters that are essential for the understanding of the algorithm.

Similarly, if we damp all eigenfrequencies, in Step 5 of Algorithm 1 we calculate approximation of the function $f(v_1,\dots,v_k;i_1,\dots,i_k)$ using the approximation algorithms introduced in [6].

Algorithm 2 (Approximation of the function $f(v_1,\dots,v_k;i_1,\dots,i_k)$)

Require: $\kappa \geq 1$; tol_{start} – tolerance for the first approximation;
ε – tolerance for bounding the approximation error;
c_1 – a positive constant for scaling the tolerance ($c_1 < 1$).
Ensure: f_{approx} – approximation of the function $f(v_1,\dots,v_k;i_1,\dots,i_k)$.
1: $tol = tol_{start}$
2: **while** $tol > 10^4$ **u do**
3: Calculate an approximation of function $f(v_1,\dots,v_k;i_1,\dots,i_k)$ with [7, Algorithm 2] using tolerance tol, and denote the approximation by f_{approx}.
4: Calculate the right-hand side of the bound given in [7, Theorem 3.1] and denote it by b
5: **if** $b < \varepsilon$ **then**
6: **return** f_{approx}
7: **break**
8: **else**
9: $tol = c_1 \cdot tol$
10: **end if**
11: **end while**

In the following example we will demonstrate the performance of damping optimization using approximation algorithms and the "discrete to continuous" approach.

Example 1. We consider the n-mass oscillator or oscillator ladder with two dampers.For such a mechanical system the mathematical model is given by $M\ddot{x} + D\dot{x} + Kx = 0$, where for the stiffness and mass matrices we have

$$K = \begin{pmatrix} k_1+k_2 & -k_2 & & & & \\ -k_2 & k_2+k_3 & -k_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -k_{n-1} & k_{n-1}+k_n & -k_n & \\ & & & -k_n & k_n+k_{n+1} \end{pmatrix}, \quad M = \operatorname{diag}(m_1,m_2,\dots,m_n),$$

where $m_i > 0$ for $i = 1\dots,n$ are the masses and $k_i > 0$ for $i = 1\dots,n+1$ are the stiffnesses. We will consider the following configuration for $n = 1600$:

$$m_i = 120-(i-1)/5, \quad i = 1,\dots,100;$$
$$m_i = i, \quad i = 101,\dots,n; \qquad k_i = 4, \quad i = 1,\dots,n+1.$$

Furthermore, we demand to damp the undamped eigenfrequencies that are smaller than 0.005 by magnitude, yielding $s = 34$ (s determines the matrix G given in (7)).

The damping matrix is $D = C_u + C_{ext}$, where the internal damping C_u is defined as in (2) with $\alpha_c = 0.001$. Since we will consider two dampers of different viscosities, the external damping is defined by $C_{ext} = v_1 e_i e_i^T + v_2 e_j e_j^T$ for $1 \leq i < j \leq n$, where e_i

is the ith canonical basis vector, and v_1, v_2 are the viscosities of the dampers applied on the ith and jth mass, respectively.

As we mentioned above, the considered optimization process is extremely demanding, because it requires solving the Lyapunov equation (9) numerous times. In order to compare the performance of our approach, instead of performing optimization in all damping positions (this corresponds to "direct" approach), we will optimize the viscosities on the following equidistant mesh of damping positions:

$$i = 51:50:n, \quad j = i+51:50:n, \tag{12}$$

which will give 465 different damping positions. We obtain that the optimal position with respect to the mesh (12), is the position $(i, j) = (651, 1352)$ with optimal viscosities $(v_1, v_2) = (107.03009, 150.49333)$, while the optimal trace is $\text{trace}(X(v_1, v_2)) = 993\,067.32851$. More details of the performance of dimension reduction for this example are shown in [7]. For the optimization of the viscosities on the above mesh we needed 104 days, which means that for viscosity optimization at one point of the mesh we need 5.4 hours on average. Recall that a "direct" approach requires the viscosity optimization for all possible configurations of damping positions. As there are $n(n-1)/2$ different positions, it is impossible to apply direct optimization here. But optimization becomes possible with our approach which combines an approximation algorithm with the "discrete to continuous" approach.

Now, we will present results obtained with the "discrete to continuous" approach presented in Algorithm 1, but in Step 5 we will use the function approximation given by Algorithm 2.

In Algorithm 1 we use the following configuration: $d_3 = 160$; $d_4 = 80$; $v_1^s = v_2^s = 50$. While in Algorithm 2 we use the configuration: $tol_{start} = 0.02$; $\varepsilon = 0.05$; $c_1 = 0.5$.

The parameters d_3 and d_4 define the grid with 45 different points in Algorithm 1. The function is minimized with the MATLAB function `fminsearch` and for a termination tolerance for the function value we take 0.1, which determines the absolute error. Thus the relative error has magnitude $\mathcal{O}(10^{-7})$. For the termination tolerance of the optimization variables we have used 0.01 (this also determines the absolute error). We have obtained that the optimal damping positions equal to $(730, 1274)$ with optimal viscosities equal to $(120.47387, 120.38917)$. For these parameters the value of our penalty function equals 987 258.34332. This value was calculated using an algorithm without dimension reduction. Note that combining Algorithm 1 with dimension reduction technique, we obtain a smaller trace. That is we have obtained the optimal positions which are not included in the mesh (12). This is not a surprise since the mesh (12) includes just a small number of positions.

For calculating the approximation of the optimal damping with this approach we only needed 0.532 days. The obtained results with corresponding CPU times were calculated using an Intel(R) Core(TM) i7 CPU 920 with 12GB of RAM and 8 MB cache.

Another possibility for optimization is the application of the heuristic introduced in [8] with the help of the approximation algorithms which use the mentioned dimension reduction techniques.

4 Conclusions

Damping optimization in a mechanical vibrating system is a very demanding problem due to the numerous Lyapunov equations which have to be solved. In this paper, we have introduced the "discrete to continuous" approach which considerably reduces the number of Lyapunov equations which have to be solved. Furthermore, we have proposed a new approach which is based on dimension reduction and continuous minimization.

We can conclude that combining the approximation algorithm with the "discrete to continuous" approach we have significantly accelerated the time needed for the calculation of the approximation of optimal damping.

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