PAMM · Proc. Appl. Math. Mech. **6**, 781–782 (2006) / **DOI** 10.1002/pamm.200610371

MPC for the Burgers Equation Based on an LQG Design

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We consider optimal control problems for semilinear parabolic PDEs where process and measurement noise can occur. We discuss the solution of such problems by using a Model Predictive Control (MPC) strategy. For the resulting sub-problems we will use a Linear Quadratic Gaussian (LQG) design. Thus we will discuss the efficient implementation of the LQG approach since it is the major computational part in the MPC scheme for this class of optimal control problems. We will present some numerical results for the Burgers equation.

1 Introduction

We consider the optimal control problem

$$\min_{u\in L^2(0,T_f;\mathcal{U})} \frac{1}{2}\int_0^{T_f} \langle y, Q\, y\rangle_{\mathcal{Y}} + \langle u, R\, u\rangle_{\mathcal{U}}\, dt + G(x(T_f)), \tag{1}$$

with $T_f \in (0,\infty]$ and $G \equiv 0$ if $T_f = \infty$, subject to the semilinear state equation

$$\dot{x}(t) \quad = \quad f(x(t)) + B\, u(t) + F\, v(t) \quad \text{for } t > 0, \quad x(0) = x_0 + \eta_0, \qquad y(t) = C\, x(t) + w(t). \tag{2}$$

Here, $x(t) \in \mathcal{X}$ (e.g., $\mathcal{X} = H_1(\Omega)$) are the states, $u(t) \in \mathcal{U}$ are the inputs and $v(t)$ is the input noise. Furthermore, $y(t) \in \mathcal{Y}$ are the outputs, $w(t)$ is the output noise and η_0 is the noise in the initial condition. We assume $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ to be Hilbert spaces. If (2) is an ODE then we have a finite-dimensional problem with $\mathcal{X} = \mathbb{R}^n, \mathcal{U} = \mathbb{R}^m, \mathcal{Y} = \mathbb{R}^p$. In the case of a PDE this is an infinite-dimensional problem. So we discretize in space (semi-discretization) to obtain an ODE.

For example, the noises can represent errors of the measurement tool or in the model. We assume that $v(t)$ and $w(t)$ are white noise stochastic processes which are uncorrelated. The (time-independent) covariance matrices of $v(t)$ and $w(t)$ are denoted by V and W.

Besides the nonlinearity, in practice we have the problem that the states may not be available because we often have no complete access to them through measurements and the occurrence of noises. So we need a strategy which is able to estimate the states. For this we will follow a strategy presented by Ito and Kunisch in [2]. The idea is to decompose the interval $[0, T_f]$ into sub-intervals and linearize the nonlinear state equation (Model Predictive Control (MPC) or Receding Horizon Control (RHC)). Then we estimate the states by using a Kalman filter. With the estimated states, we compute a control law on the sub-interval based on a Linear Quadratic Gaussian Design (LQG).

2 MPC/LQG Design

The first step is a decomposition of the interval $[0, T_f]$ into sub-intervals $[T_i, T_i + T]$. Then we consider the optimal control problem (1) – (2) on $[T_i, T_i + T]$. For the linearization on the sub-interval we need a reference pair $(x^*(t), u^*(t))$ which is known or has to be computed. We linearize (2) around $(x^*(t), u^*(t))$ and subsequently replace $x^*(t)$ by an operating point $\bar{x}$ which can be determined for example as

$$\bar{x} = \frac{1}{T}\int_{T_i}^{T_i+T} x^*(t)\, dt.$$

So we obtain after resorting

$$\dot{x}(t) = A(x(t) - x^*(t)) + f(x^*(t)) + Bu(t) + Fv(t), \tag{3}$$

where $A = f'(\bar{x})$. The optimal control is given by a feedback law (see [2]). Applied to our problem this results in

$$u(t) = u^*(t) - K(\hat{x}(t) - x^*(t)) \tag{4}$$

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where $K = R^{-1}B^T X_*$ with X_* being the solution of the Algebraic Riccati Equation (ARE)

$$0 = X\,A + A^T X - X\,BR^{-1}B^T X + C^T QC, \tag{5}$$

and $\hat{x}(t)$ is an estimation of the state $x(t)$. The estimated state is provided by the Kalman-Bucy filter (see [3, 5]) which results in

$$\dot{\hat{x}}(t) = A(\hat{x}(t) - x^*(t)) + f(x^*(t)) + Bu(t) + L\,(C(x(t) - \hat{x}(t)) + w(t))\,, \quad \hat{x}(0) = x_0 + \eta_0, \tag{6}$$

where $L = \Sigma_* C^T W^{-1}$ and Σ_* is the solution of the Filter Algebraic Riccati Equation (FARE)

$$A\,\Sigma + \Sigma\,A^T - \Sigma\,C^T W^{-1} C\,\Sigma + FVF^T = 0. \tag{7}$$

So the effort on every sub-interval $[T_i, T_i + T]$ is the following:

(I) Determine a reference pair (x^*, u^*). (II) Linearize the state equation. (III) Solve the AREs (5) and (7) (see [1, 4]) to obtain the gain matrices K and L. (IV) Solve (6) by using (3) for the measurements. (V) Compute the optimal control (4).

Afterwards we move to the next sub-interval $[T_{i+1}, T_{i+1} + T]$ and repeat the whole procedure.

3 Numerical Example

Here we demonstrate the performance of the MPC/LQG scheme applied to the Burgers equation

$$x_t(t,\xi) + x(t,\xi)\,x_\xi(t,\xi) = \nu x_{\xi\xi}(t,\xi), \tag{8}$$

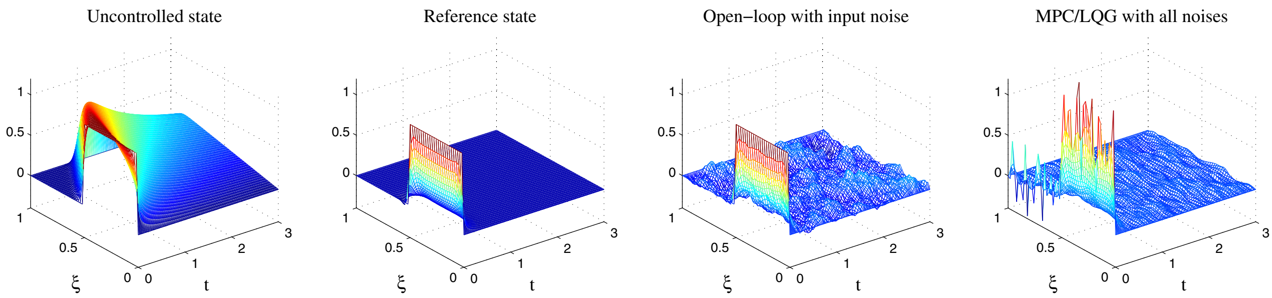
where t is the variable in time, ξ the variable in space, and ν is a viscosity parameter. So we consider an optimal control problem of the form (1) subject to the Burgers equation

$$\begin{aligned} x_t(t,\xi) &= \nu\,x_{\xi\xi}(t,\xi) - x(t,\xi)\,x_\xi(t,\xi) + B(\xi)u(t) + F(\xi)v(t), \qquad y(t,\xi) = C\,x(t,\xi) + w(t,\xi), \\ x(t,0) &= x(t,1) = 0, \quad t > 0, \qquad x(0,\xi) = x_0(\xi) + \eta_0(\xi), \quad \xi \in (0,1). \end{aligned} \tag{9}$$

If we discretize (9) in space by using finite elements we obtain an ODE of the form (2). We use the following parameters

$$T_f = 3, T = 0.5, dt = d\xi = \frac{1}{50}, \nu = 0.01, C = \mathrm{I}, B = F = 1_{\Omega_u}(\xi), Q = 0.1\mathrm{I}, R = 0.001\mathrm{I}, V = 4\mathrm{I}, W = 0.01\mathrm{I}.$$

For the noises we choose normally distributed random numbers with standard deviations of 2 and 0.1 for $v(t)$ and $w(t)$, respectively. Because of the choice of T_f and T we have six sub-intervals.



The initial condition is one on $(0, 0.5]$ and zero on $(0.5, 1)$. The first figure shows the uncontrolled solution of (8). The reference trajectory in the second figure was computed by solving a two-point boundary value problem without noises. Figure 3 is an open-loop solution with input noise and the last figure is the state of the MPC/LQG solution with noise in input, output and initial condition which is much smoother than in Figure 3.

The figures demonstrate that the MPC/LQG scheme is able to control the solution of the Burgers equation to zero in the presence of noise in inputs, outputs and initial condition.

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