MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

MINIMAL REALIZATION AND MODEL REDUCTION OF STRUCTURED SYSTEMS

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DFG-Graduiertenkolleg
MATHEMATISCHE
KOMPLEXITÄTSREDUKTION

Overview

Outline

Original System ($E = I_n$)

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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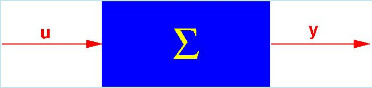

Goals:

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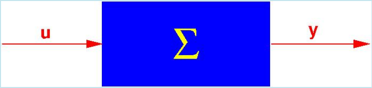

Reduced-Order Model (ROM)

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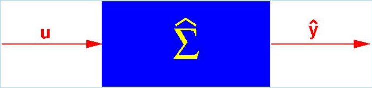

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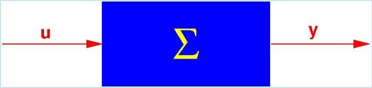

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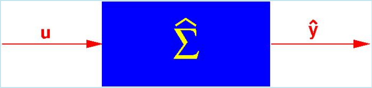

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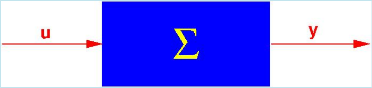

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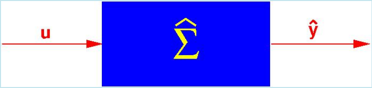

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$||y - \hat{y}|| <$ tolerance $\cdot\ ||u||$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s) - x(0))$ to LTI system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

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$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:\mathbf{H}(s)}\Big)u(s).$$

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Model reduction in frequency domain: Fast evaluation of mapping $u \ \rightarrow \ y$.

Formulating model reduction in frequency domain

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n},\ D \in \mathbb{R}^{p\times m}, \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \quad \hat{A} \in \mathbb{R}^{r\times r},\ \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{p\times r},\ \hat{D} \in \mathbb{R}^{p\times m} \end{aligned}$$

of order $r \ll n$, such that

$$||y - \hat{y}|| = \left|\left|\mathbf{H}u - \hat{\mathbf{H}}u\right|\right| \leq \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right| \cdot ||u|| < \text{tolerance} \cdot ||u|| \,.$$

$\Longrightarrow$ Approximation problem: $\displaystyle\min_{\text{order}\,(\hat{\mathbf{H}})\leq r} \left|\left|\mathbf{H} - \hat{\mathbf{H}}\right|\right|$,

where, mostly, $\|\,.\,\| = \|.\|_{\mathcal{H}_\infty}$ or $\|\,.\,\| = \|\,.\,\|_{\mathcal{H}_2}$.

Second-order / mechanical / vibrational systems:

$$M\ddot{x}(t) + L\dot{x}(t) + Kx(t) = Bu(t), \qquad y(t) = C_p x(t) + C_v \dot{x}(t).$$

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Apply Laplace transform $\rightsquigarrow$

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Other examples: integro-differential / fractional systems, systems with surface loss, 1D PDE control, . . . Note: all systems are linear w.r.t. the mapping $u \to y$!

Consider **Structured Linear System (SLS)** in frequency domain, using general set-up:

$$\boxed{\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),} \tag{1}$$

where

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4) **EM w/ surface loss:** $\mathcal{C}(s) = sB$, $\mathcal{B}(s) = B$, and $\mathcal{K}(s) = s^2M + sL + K - \frac{1}{\sqrt{s}}N$.

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5) **Integro-differential Volterra systems**, **input delays**, **fractional systems** ...

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find projection matrices

$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n\times r}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}_r,$$

(with $r \ll n$), such that

$$\hat{\mathbf{H}}(s) = \hat{\mathcal{C}}(s)\hat{\mathcal{K}}(s)^{-1}\hat{\mathcal{B}}(s), \text{ where}$$

Given a large-scale **SLS**

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s),$$

find projection matrices

$$\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}, \quad \mathbf{W}^T\mathbf{V} = \mathbf{I}_r,$$

(with $r \ll n$), such that

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- Note $\hat{\mathbf{A}}_i = \mathbf{W}^T\mathbf{A}_i\mathbf{V}$, $\hat{\mathbf{E}} = \mathbf{W}^T\mathbf{E}\mathbf{V}, \hat{\mathbf{C}}_i = \mathbf{C}_i\mathbf{V}$ and $\hat{\mathbf{B}}_i = \mathbf{W}^T\mathbf{B}_i$.
- The ROM preserves the $\alpha_i(s), \beta_i(s)$ and $\gamma_i(s)$ functions.

Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction.

[Beattie/Gugercin '09]

Interpolation points $\sigma_k,\ \mu_j$ $\Rightarrow$ $\boxed{\begin{array}{l}\mathcal{K}^{-1}(\sigma_k)\mathcal{B}(\sigma_k) \in \operatorname{range}(\mathbf{V}) \ \text{ and} \\ \mathcal{K}^{-T}(\mu_k)\mathcal{C}^T(\mu_j) \in \operatorname{range}(\mathbf{W}).\end{array}}$

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Balancing truncation methods

- Structure-preserving model reduction for integro-differential equations. [Breiten '16]

$$\mathbf{P} = \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{K}_s(s)^{-1}\mathcal{B}(s)\mathcal{B}(s)^T\mathcal{K}(s)^{-T}ds,$$
$$\mathbf{Q} = \frac{1}{2\pi}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \mathcal{K}_s(s)^{-T}\mathcal{C}(s)^T\mathcal{C}(s)\mathcal{K}(s)^{-1}ds.$$

$\Rightarrow$ Find $\mathbf{V}, \mathbf{W}$ from $T^{-1}PQT = \Sigma$.

Interpolation-based methods

- Interpolatory projection methods for structure-preserving model reduction. [Beattie/Gugercin '09]

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Data-driven methods

- Data-driven structured realization. [Schulze/Unger/Beattie/Gugercin '18]

Outline

Let us consider the first order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{A} = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & -1 \\ 0 & 0 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \text{ and } \mathbf{C}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

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Note that $\mathbf{H}(s) = \dfrac{1}{s+2} = \hat{\mathbf{H}}(s) = \hat{\mathbf{C}}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}}$, with $\hat{\mathbf{A}} = -2, \hat{\mathbf{B}} = 1$ and $\hat{\mathbf{C}} = 1$.

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Minimal realization problem

Find order r and matrices $\mathbf{V}$ and $\mathbf{W}$ such that the reduced-order model obtained by projection satisfies

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Solutions:

- Kalman reachability/observability criteria,
- Hankel matrix (Silverman method),
- reachability and observability Gramians,
- **Loewner matrix.** [Mayo/Antoulas '07]

For illustration, consider the time-delay systems

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Minimal realization problem

Is there a way to find the order r and matrices $\mathbf{V}, \mathbf{W} \in \mathbb{R}^{n \times r}$ such that the system $\hat{\mathbf{H}}(s)$ obtained by projection is "minimal", *i.e*

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s?$$

Given a first order system

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}, \text{ with } \mathbf{E} \in \mathbb{R}^{n \times n} \text{ invertible.}$$

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Reachability characterization [Anderson/Antoulas '90]

If $(\mathbf{E}, \mathbf{A}, \mathbf{B})$ is $\mathbb{R}^n$-reachable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

$$\mathbf{R} = \begin{bmatrix} (\sigma_1\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} & \dots & (\sigma_t\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \end{bmatrix}. \text{ Then } \operatorname{rank}(\mathbf{R}) = n.$$

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If $(\mathbf{E}, \mathbf{A}, \mathbf{C})$ is $\mathbb{R}^n$-observable, $t \geq n$, $\sigma_i \neq \sigma_j$ for $i \neq j$, and

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Rank encodes minimality [Anderson/Antoulas '90]

$$\operatorname{rank}\left(\mathbf{O}^T\mathbf{E}\mathbf{R}\right) = \text{order of minimal realization} = \text{r}.$$

For **SLS**, we use the notion of $\mathbb{R}^n$ reachability and observability. Let us consider the SLS

$$\mathbf{H}(s) = \mathcal{C}(s)\mathcal{K}(s)^{-1}\mathcal{B}(s) \text{ of order n.}$$

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Rank encodes minimality

$$\operatorname{rank}\left(\mathbf{O}^T\mathbf{E}\mathbf{R}\right) = \text{order of the } \textbf{SLS} \text{ "minimal" realization} = r.$$

Let's go back to the time-delay example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1}\mathbf{B}, \text{ with } \mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

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So, we get the projection matrices

$$\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1:2) \text{ and } \mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1:2).$$

Let's go back to the time-delay example

$$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A}_1 - \mathbf{A}_2 e^{-s})^{-1}\mathbf{B}, \text{ with } \mathbf{A}_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{A}_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \text{ and } \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

Let us construct, for $\sigma_i = [1, 2, 3, 4, 5]$,

$$\mathbf{R} = \begin{bmatrix} K(\sigma_1)^{-1}\mathbf{B} & \ldots & K(\sigma_5)^{-1}\mathbf{B} \end{bmatrix},$$
$$\mathbf{O} = \begin{bmatrix} K(\sigma_1)^{-T}\mathbf{C}^T & \ldots & K(\sigma_5)^{-T}\mathbf{C}^T \end{bmatrix}.$$

Hence, we see that

- $\operatorname{rank}(\mathbf{R}) = \operatorname{rank}(\mathbf{O}) = 2.$ (nonreachable nonobservable)
- $\operatorname{rank}(\mathbf{O}^T\mathbf{R}) = 2.$ (minimal realization order)

Then,

$$[\mathbf{Y}, \Sigma, \mathbf{X}] = \textsf{svd}(\mathbf{O}^T\mathbf{R}).$$

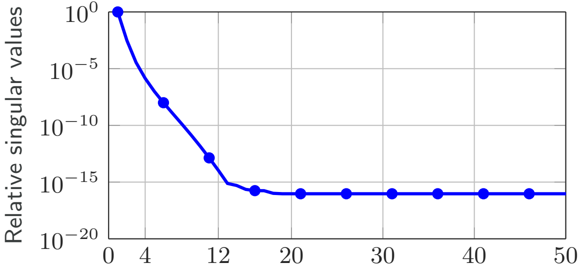
So, we get the projection matrices

$$\mathbf{V} = \mathbf{R}\mathbf{X}(:, 1:2) \text{ and } \mathbf{W} = \mathbf{O}\mathbf{Y}(:, 1:2).$$

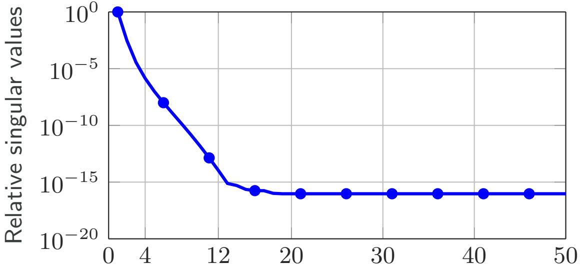
The $\hat{\mathbf{H}}$ obtained using $\mathbf{V}$ and $\mathbf{W}$ satisfies

$$\mathbf{H}(s) = \hat{\mathbf{H}}(s), \forall s.$$

Outline



- Figure represents the singular values of $\mathbf{O}^T\mathbf{E}\mathbf{R}$ for a large-scale time-delay example.



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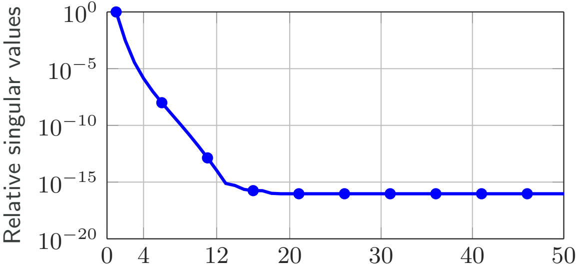

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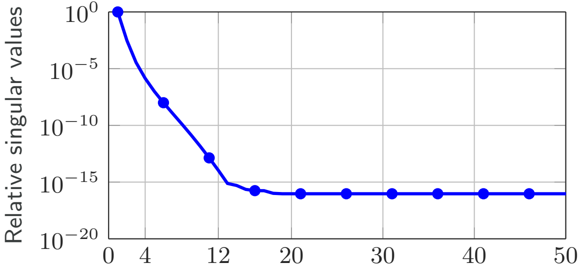

- Figure represents the singular values of $\mathbf{O}^T\mathbf{ER}$ for a large-scale time-delay example.
- For large-scale systems, often low-rank phenomena can be observed.
- Numerical rank of $\mathbf{O}^T\mathbf{ER}$ generally small compared to n.
- We can cut off states that are related to very small singular value of $\mathbf{O}^T\mathbf{ER}$.

To compute $\mathbf{R}$ (analogously for $\mathbf{O}$),

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$$\boxed{\mathbf{ERS} - \sum_{i=1}^{\ell_\alpha} \mathbf{A}_i \mathbf{R} \mathbf{M}_i = \sum_{i=1}^{m} \mathbf{B}_i \mathbf{b}_i,}$$
where
$$\begin{aligned} \mathbf{M}_i &= \operatorname{diag}\left(\alpha_i(\sigma_1), \ldots, \alpha_i(\sigma_t)\right) \\ \mathbf{b}_i &= \left[\beta_i(\sigma_1), \ldots, \beta_i(\sigma_t)\right], \\ \mathbf{S} &= \operatorname{diag}\left(\sigma_1, \ldots, \sigma_t\right). \end{aligned}$$

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- We use the truncated low-rank methods for generalized Sylvester equations from [Kressner/Sirkovic '15].

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Input: SLS $\mathcal{K}(s)$, $\mathcal{B}(s)$, $\mathcal{C}(s)$ and reduced order r.

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Output: Reduced-order model is given by

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Outline

1. Introduction
2. Minimal Realization
3. Reachability and Observability for SLS
4. Model Order Reduction
5. Numerical Results
   - A Time-delay System
   - Second-order System
   - Parametric Systems
   - Fitz-Hugh Nagumo Model
6. Outlook and Conclusions

Let us consider the time delay system

$$\dot{x}(t) = Ax(t) + A_\tau x(t-\tau) + Bu(t),$$
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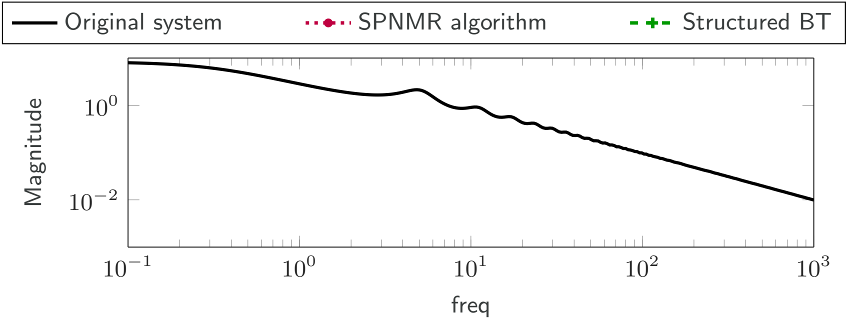
- Heated rod cooled using delayed feedback from [Breda/Maset/Vermiglio '09].
- Full order model $n = 120$ and $\tau = 1$.
- ROM obtained used SPNMR method (100 log. dist. points in $[1e^{-1}, 1e^{3}]i$) and Structured Balanced Truncation [Breiten '16].
- Reduced order $r = 4$.



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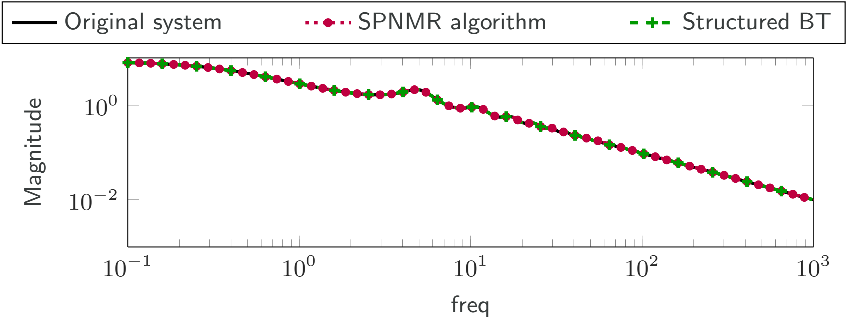

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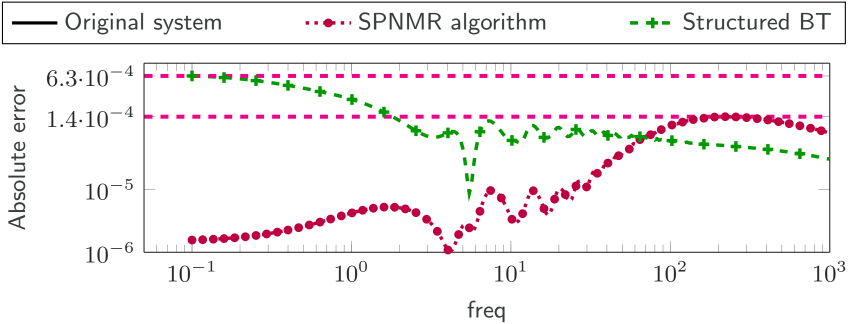

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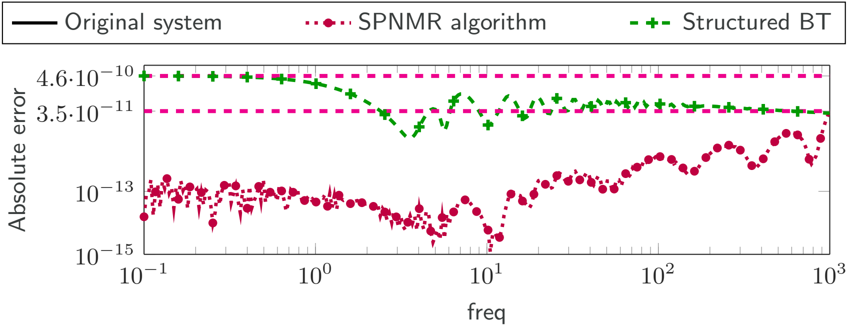

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$$M\ddot{x}(t) + D\dot{x}(t) + Kx(t) = Bu(t)$$
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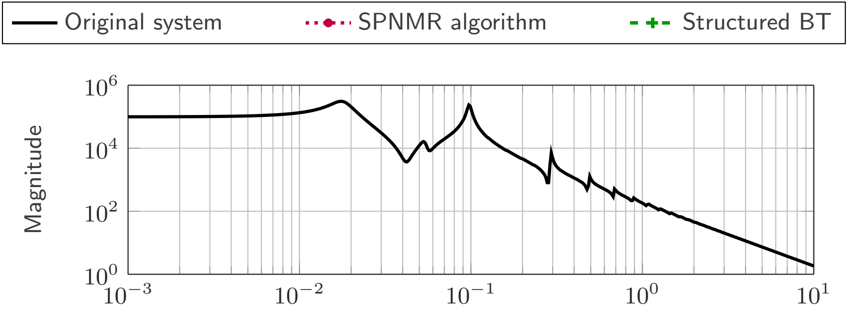

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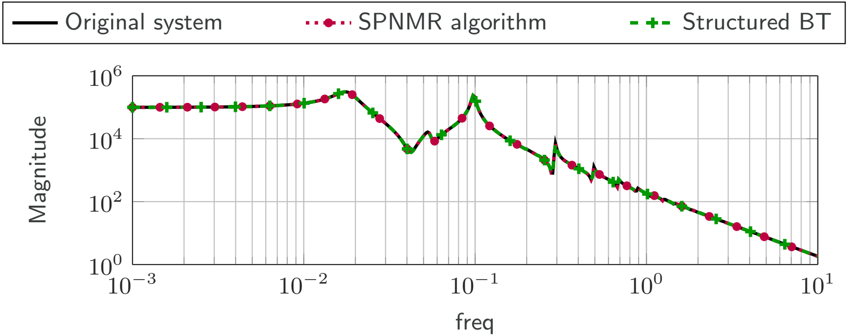

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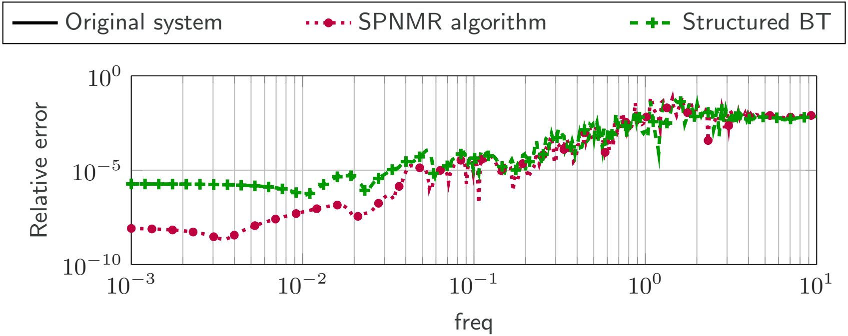

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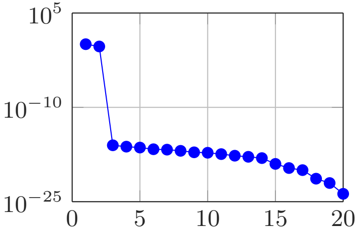
- Compute projectors $\mathbf{V}$ and $\mathbf{W}$ and $\hat{\mathbf{H}}(s,p)$.
- Then, $\mathbf{H}(s,p) = \hat{\mathbf{H}}(s,p)$.

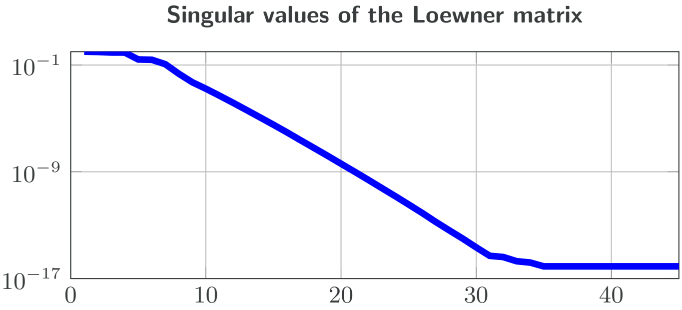
Decay of Singular values



- FOM example [MORWIKI][1] of order 1006 and $p \in [10, 100]$ of the form

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_1 + p\mathbf{A}_2)\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- 1500 random points $(s, p) \in [1e0, 1e4]i \times [10, 100]$. Reduced order $r = 15$.



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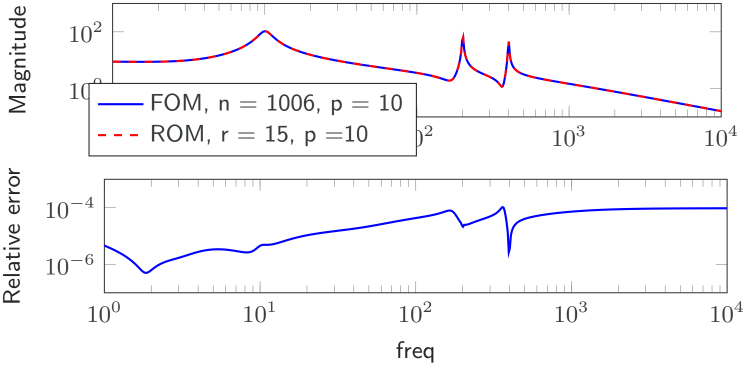

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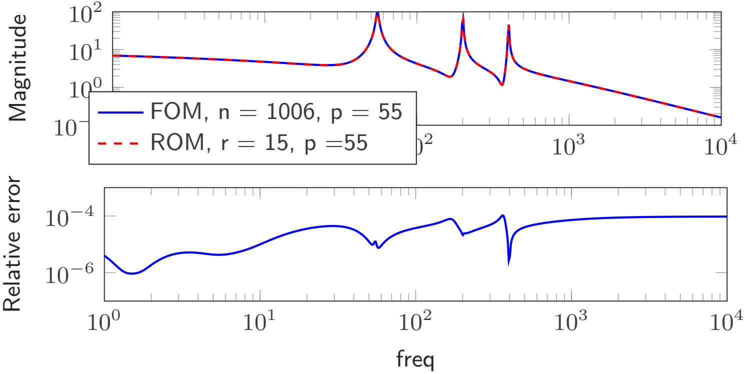

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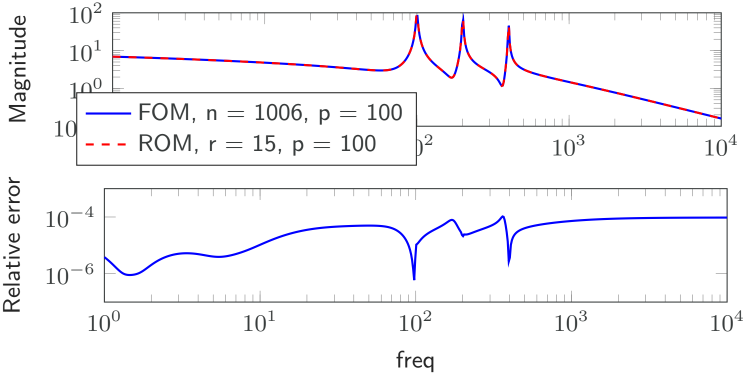

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- Consider again the FOM model [MORWIKI][2] of order 1006 and $p \in [10, 100]$ with an artificial delay ($\tau = 3s$)

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + p\mathbf{A}_2\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t)$$
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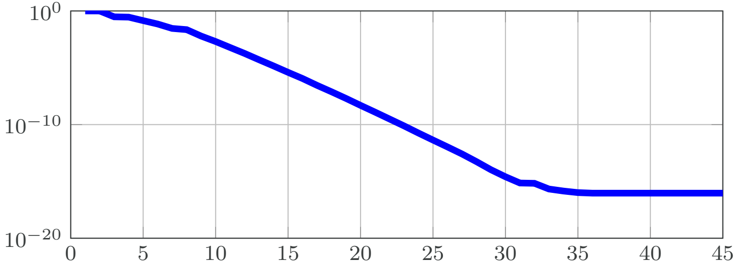
Singular values of the Loewner matrix



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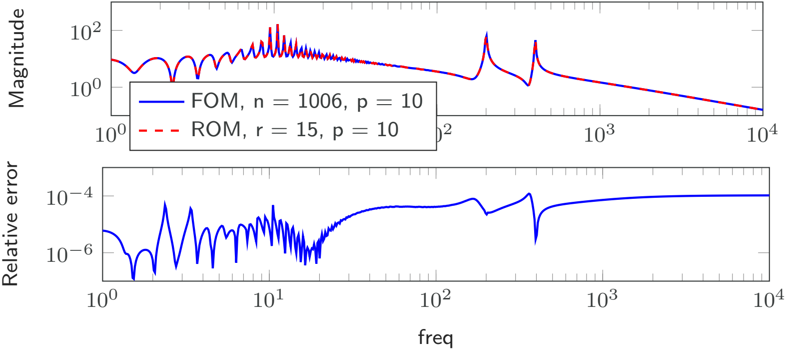

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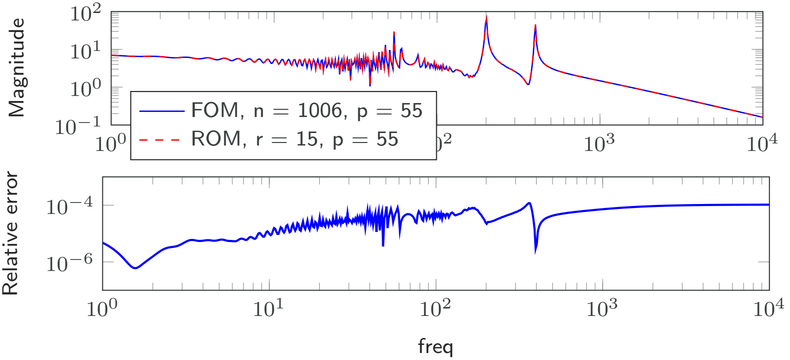
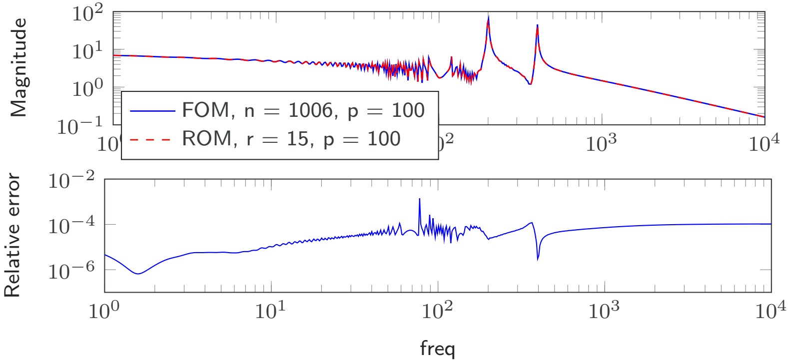


[2] morwiki.mpi-magdeburg.mpg.de/

- Consider again the FOM model [MORWIKI][2] of order 1006 and $p \in [10, 100]$ with an artificial delay ($\tau = 3s$)

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + p\mathbf{A}_2\mathbf{x}(t-\tau) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

- 1500 randomly chosen points $(s, p) \in [1e0, 1e4]i \times [10, 100]$. Reduced order $r = 15$.
- p $=$100



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Fitz-Hugh Nagumo model: Governing coupled equation

$$\begin{aligned} \epsilon v_t &= \epsilon^2 v_{xx} + v(v-0.1)(1-v) - w + u, \\ w_t &= hv - \gamma w + u \end{aligned} \qquad \text{on} \quad [0,T] \times [0,L]$$

with initial and boundary conditions

$$v(x,0) = 0, \quad w(x,0) = 0, \quad x \in (0,L), \qquad v_x(0,t) = i_0(t), \quad v_x(L,t) = 0, \quad t \geq 0.$$

- To employ the interpolation-based algorithm, we choose random 100 interpolation points in a logarithmic way between $[10^{-2}, 10^2]$ and set $\sigma_i = \mu_i$, $i \in \{1, \ldots, 100\}$.

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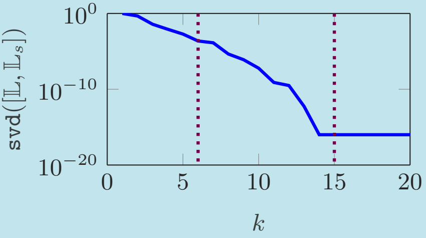
Decay of singular values of Loewner pencil



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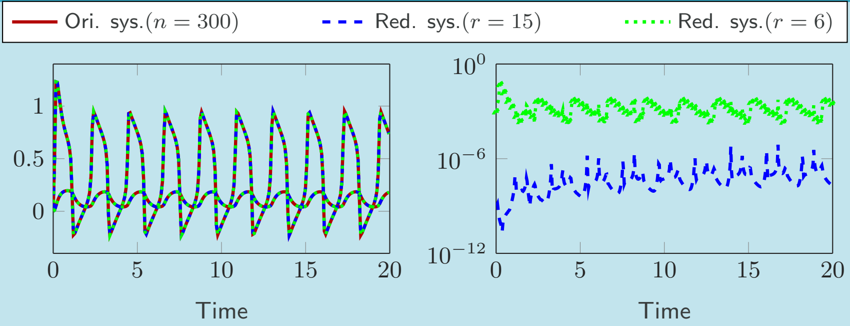
Construction of reduced systems



Contribution of this talk

- Minimal realization by projection of **SLS**.
- Model reduction technique inspired by numerical rank of matrix $\mathbf{O}^T\mathbf{E}\mathbf{R}$.
- Projector computation solving generalized Sylvester equation (low-rank methods).
- Performance illustrated by numerical examples for several system classes.
- Extended results to parametric **SLS**.

Outlook and Conclusions

Contribution of this talk

- Minimal realization by projection of **SLS**.
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- Performance illustrated by numerical examples for several system classes.
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Open questions and future work

- Stability preservation and error bounds.
- Relation to pure Loewner-style approach [Schulze/Unger/Beattie/Gugercin '18]?
- Extension to nonlinear systems, first results for polynomial systems in [Benner/Goyal '19, arXiv:1904.11891].