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The Riccati Eigenproblem

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Overview

Motivation

The LQR Problem

The linear-quadratic regulator (LQR) problem

$$\min_{u \in L_2[0,\infty]} \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t)\, dt \quad (= \mathcal{V}(x_0)) \tag{1}$$

subject to

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathbb{R}^n. \tag{2}$$

$A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $0 \le Q = Q^T \in \mathbb{R}^{n\times n}$, $0 < R = R^T \in \mathbb{R}^{m\times m}$.

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Solution: optimal control/Riccati feedback [Kalman 1960]

$$u_*(t) = -R^{-1} B^T X_* x(t), \tag{3}$$

where $X_* = X_*^T \in \mathbb{R}^{n \times n}$ is the unique positive semidefinite solution of the algebraic Riccati equation (ARE)

$$0 = Q + A^T X + XA - XBB^T X =: \mathcal{R}(X). \tag{4}$$

Motivation

The LQR Problem

Properties of the Riccati solution

$$0 = Q + A^T X + XA - XBB^T X =: \mathcal{R}(X).$$

(I) X_* is unique element in $\mathbb{X} := \{ X \in \mathbb{R}^{n \times n} \,|\, \mathcal{R}(X) = 0 \}$ which is spsd and spd if (A, Q) controllable.

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(III) X_* is unique element in $\mathbb{X}$ with

$$\Lambda\left(A - BR^{-1}B^T X\right) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,|\, \Re(z) < 0\}$$

$$\Longrightarrow \quad \lim_{t\to\infty} x(t; u_*) = 0.$$

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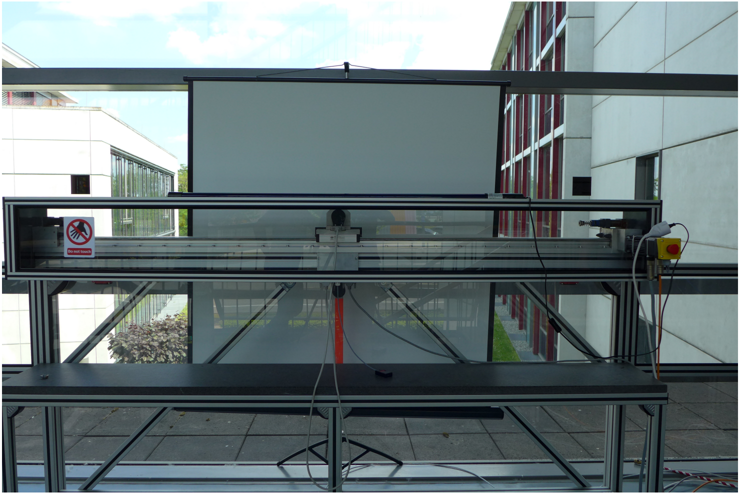
(IV) $\mathcal{V}(x_0) = x_0^T X_* x_0.$

(V) $u_*(t) = \mathrm{argmin}_{u(t)} \frac{d}{dt}\|x(t)\|_{X_*}$ pointwise in t.

LQR Control in Action

An example: the double inverted pendulum



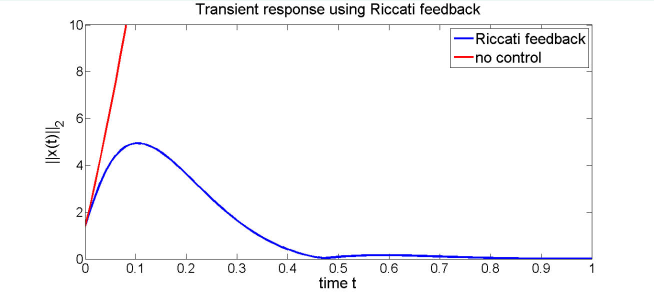
Movie thanks to Patrick Kürschner, Chris Miller; 3 June 2014.

Motivation

Goal: Avoid overshoot!

Avoid/reduce large increase in $\|x(t)\|_2$ in transient phase ("overshoot").

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = Q = R \quad \rightsquigarrow \quad X = \begin{bmatrix} 0.1628 & 1.1398 \\ 1.1398 & 16.1214 \end{bmatrix}.$$



Ideally, guarantee monotonic decrease of $\|x(t)\|$ from x_0!

The Riccati Eigenproblem

Problem statement

Motivated by the Lyapunov eigenproblem [KOHAUPT 2008]

$$A^T P + PA = \lambda P,$$

and the improved stability results for free dynamical systems $\dot{x} = Ax$ using $\| \,.\, \|_P$ instead of $\| \,.\, \|_2$, we investigate the

Riccati eigenproblem (RICEP)

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where for the LQR problem, $G = BR^{-1}B^T$.
We aim at eigenvalues $\lambda \in \mathbb{R}$ with $\lambda < 0$ and "eigenvectors" $X > 0$.

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Q: can the transient behavior of closed-loop system be improved by replacing ARE solution by appropriate eigenvector of RICEP?

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Q: can the transient behavior of closed-loop system be improved by replacing ARE solution by appropriate eigenvector of RICEP?
Note: formally, we replace constant term Q in ARE by $-\lambda X$ in RECIP. This corresponds to modified cost functional

$$\min_{u \in L_2[0,\infty]} \int_0^\infty -\lambda x(t)^T X x(t) + u(t)^T R u(t) \, dt.$$

The Riccati Eigenproblem

Results

Theorem

For $G = G^T \geq 0$, consider RICEP

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Suppose $\lambda < 0$, $X > 0$ is an eigenpair of RICEP, then:

a) *X is stabilizing, i.e., $\Lambda(A - GX) \subset \mathbb{C}^-$.*

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b) *Consider the minimization problem*

$$\min_{u(t)} \frac{d}{dt} \|x(t)\|_X \quad \textit{pointwise in } t.$$

Its optimal solution is $u_*(t) = -R^{-1}B^T X x(t)$.

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Suppose $\lambda < 0$, $X > 0$ *is an eigenpair of RICEP, then:*

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Its optimal solution is $u_*(t) = -R^{-1}B^T X x(t)$.

c) *It holds*

$$\frac{d}{dt}\|x(t)\|_X \leq \rho \|x(t)\|_X.$$

$\Rightarrow$ *Strict mon. decrease of* $\|x(t)\|_X \searrow 0 \rightsquigarrow$ *vibration suppression.*

The Riccati Eigenproblem

A numerical algorithm

Lyapunov Eigenvalue Iteration

INPUT. $A \in \mathbb{R}^{n\times n}$, $0 \leq G = G^T \in \mathbb{R}^{n\times n}$, X_0 with $\Lambda(A - \frac{1}{2}GX_0) \subset \mathbb{C}^-$.

OUTPUT. Riccati eigenpair (λ, X) with $\lambda < 0$, $X = X^T \geq 0$.

FOR $j = 1, \ldots$

Solve Lyapunov eigenproblem

$$(A - \frac{1}{2}GX_j)^T X_{j+1} + X_{j+1}(A - \frac{1}{2}GX_j) = \lambda_j X_{j+1}$$

for minimal real eigenvalue.

ENDFOR

In case of convergence, $X_j \to X$, $\lambda_j \to \lambda$.

The Riccati Eigenproblem

Properties of Lyapunov eigenvalue iteration

- Partial convergence results.

The Riccati Eigenproblem

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- Lyapunov eigenvalue problem is a standard matrix eigenvalue problem:

$$\left((A - \frac{1}{2} G X_j) \otimes I_n + I_n \otimes (A - \frac{1}{2} G X_j) \right) x_{j+1} = \lambda_j x_{j+1},$$

where $x_j = \mathrm{vec}\,(X_j)$.
Note: there exist at least n negative real eigenvalues $\lambda_k(A) + \overline{\lambda_k(A)}$.

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  where $x_j = \mathrm{vec}\,(X_j)$.
  Note: there exist at least n negative real eigenvalues $\lambda_k(A) + \overline{\lambda_k(A)}$.
- Alternative: solve non-homogeneous Lyapunov eigenproblem:

$$(A - GX_j)^T X_{j+1} + X_{j+1}(A - GX_j) = \lambda_j X_{j+1} - X_j GX_j.$$

The Riccati Eigenproblem

Alternative Riccati-like eigenproblems

1. Replacing Q in LQR cost functional by $\lambda X + XGX \Longrightarrow$

$$(A - GX)^T X + X(A - GX) = \lambda X.$$

Advantages:

- $\frac{d}{dt}\|x(t)\|_X = \rho\|x(t)\|_X$.
- More natural Lyapunov eigenvalue iteration.

The Riccati Eigenproblem

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   Advantages:
   - $\frac{d}{dt}\|x(t)\|_X = \rho\|x(t)\|_X$.
   - More natural Lyapunov eigenvalue iteration.

2. Lyapunov eigenvalue problem naturally has rank-1 eigenvectors $P_k = y_k y_k^H$, where y_k are eigenvectors of A. $\rightsquigarrow$ Consider the coupled nonlinear EVP:

$$\begin{aligned} (A - GX)^T X_k + X_k(A - GX) &= \lambda_k X_k, \quad k = 1, \ldots, n, \\ X &= \sum_{k=1}^{n} X_k. \end{aligned}$$

   We have a Lyapunov eigenvalue style iteration for this.

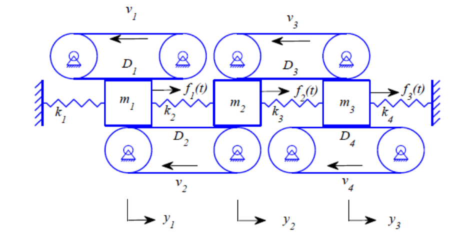
The Riccati Eigenproblem

Example

Model

modified from [Hagedorn 1978]

ℓ-DOF model with Coulomb-like friction force functions D_k, generated by conveyors moving with velocities v_k:



Linearization $\rightsquigarrow$ 2nd order linear dynamical system $M\ddot{y} + S\dot{y} + Ky = Fu \iff$

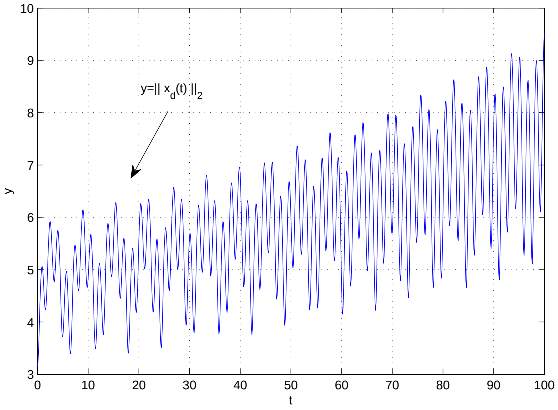
$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}S \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1}F \end{bmatrix} u,$$

with $\Lambda(A) \subset \{z \,|\, \Re(z) > 0\}$.

The Riccati Eigenproblem

Example: coupled conveyor belts

Open-loop dynamics



The Riccati Eigenproblem

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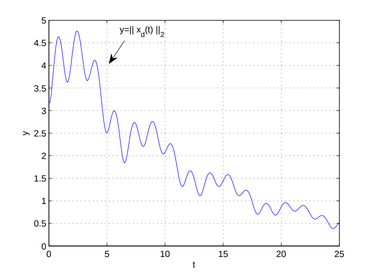
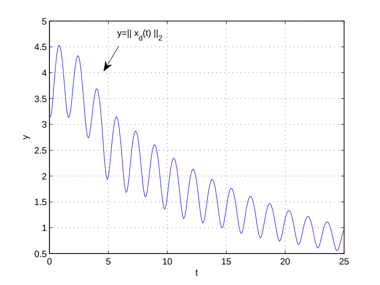
Closed-loop dynamics measured in $\| . \|_2$



Feedback via ARE



Feedback via RICEP

The Riccati Eigenproblem

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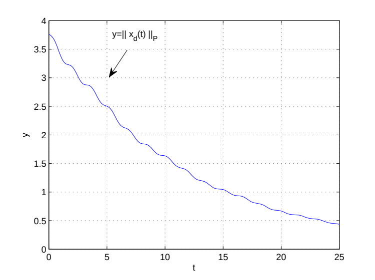
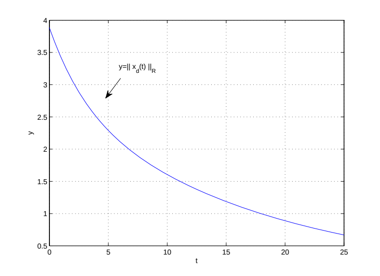
Closed-loop dynamics measured in $\|.\|_X$



Feedback via ARE



Feedback via RICEP

Conclusions and Outlook

- The Riccati EVP is a nonlinear EVP with linear dependence on the eigenvalue, but nonlinear in the eigenvector.
- Structure of spectrum and geometry of eigenspaces widely unknown.
- Solutions corresponding to real negative eigenvalues can be used in feedback control instead of the classical LQR solution.
- Effects of overshoot in the transient phase can be avoided or reduced, vibration suppression in some cases.
- Which eigenpair provides best feedback controller?
- Better numerical methods for the Riccati EVP are required.

Further reading:

P. Benner and L. Kohaupt.
On a nonlinear Riccati matrix eigenproblem.
MPI Magdeburg Preprint MPIMD/14-?, 2014 (upcoming).

L. Kohaupt.
Solution of the matrix eigenvalue problem $VA + A^*V = \mu V$ with applications to the study of free linear systems.
Journal of Computational and Applied Mathematics 213(1):142-165, 2008.