MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC
COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

20 YEARS
1998-2018

System-Theoretic Model Reduction for Nonlinear (Parametric) Systems

Peter Benner

Joint work with
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Minisymposium: Recent Advancements in Model Reduction for Stochastic and Nonlinear Systems

ICIAM, Valencia, Spain
June 19, 2019

Outine

Physical systems

Physical systems

Circuits laws, mechanics, etc.

Partial differential equations

e.g., Navier-Stokes equations

$$\dot{v} + (v \cdot \nabla)v - \frac{1}{\mathsf{Re}}\Delta v + \nabla p = 0,$$
$$\nabla \cdot v = 0.$$

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Using expert knowledge, experimental data

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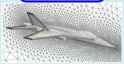

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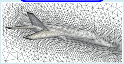

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–Reduced-order modeling motivation–

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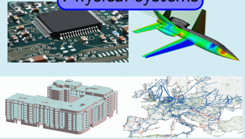

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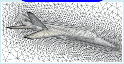

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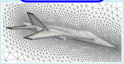

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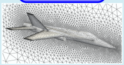

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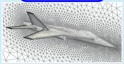

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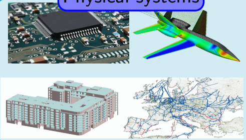

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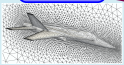

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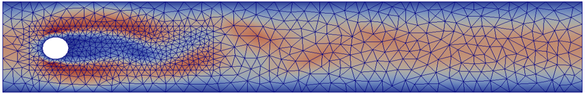
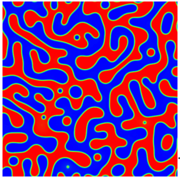
- It is possible to rewrite a large class of nonlinear systems into polynomial systems,
- via McCormick Relaxation $\rightsquigarrow$ **no approximation**. [McCormick '76, Gu '09]

Applications

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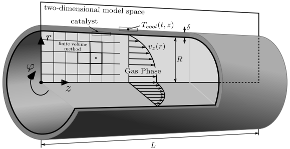

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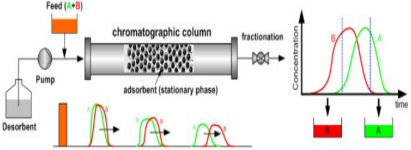
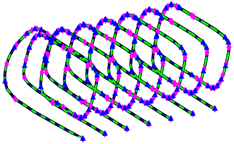


Batch Chromatography



††Courtesy of [Hawick/Playne '10]

Transformation of Nonlinear Systems

A nonlinear system

$$\dot{x}_1(t) = -x_1(t) + x_2^3(t) + e^{-x_2(t)}, \tag{1a}$$

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\dot{z}_1(t) &= x_1(t)z_1(t) - z_1(t)u(t).
\end{aligned}$$

Construction of ROMs

Full-order system

$$E\dot{x}(t) = Ax(t) + \sum_{\xi=2}^{d} H_\xi x^{\textcircled{\xi}}(t)(t) + \sum_{\eta=1}^{d} N_\eta \left(u(t) \otimes x^{\textcircled{\eta}}(t)\right) + Bu(t),$$
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Petrov-Galerkin projection

Reduced-order system

$$\hat{E}\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \sum_{\xi=2}^{d} \hat{H}_\xi \hat{x}^{\textcircled{\xi}}(t)(t) + \sum_{\eta=1}^{d} \hat{N}_\eta \left(u(t) \otimes \hat{x}^{\textcircled{\eta}}(t)\right) + \hat{B}u(t),$$
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$$\hat{E} = \mathbf{W}^T E V, \quad \hat{A} = \mathbf{W}^T A\mathbf{V}, \quad \hat{H}_\xi = \mathbf{W}^T H_\xi \mathbf{V}^{\textcircled{\xi}}, \quad \xi \in \{2, \ldots, d\},$$
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Existing Approaches

Snapshot-based methods

- Proper orthogonal decomposition, e.g., [Volkwein '08]
- Reduced basis methods, e.g., [Quarteroni et al. '16]
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  - Balanced truncation [B./Goyal '17]
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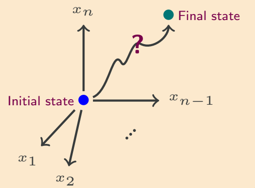

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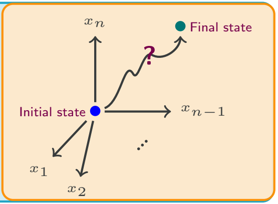

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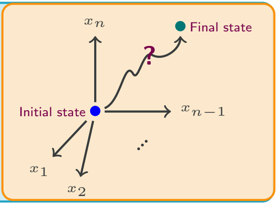

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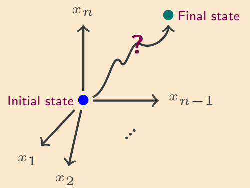

Idea of balanced truncation

- Construct state transformation, allowing to find states which are hard to reach, as well as hard to observe.
- Truncating such states yields a reduced-order system.

For linear systems

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \; x(0) = 0. \end{cases}$$

- Map between $u(t) \mapsto x(t)$:

$$x(t) = \int_0^t e^{A\sigma} B \, u(t-\sigma) d\sigma.$$

Reachability Gramian:

$$P := \int_0^{+\infty} e^{At} B (e^{At} B)^T dt.$$

- Map between $x(t) \mapsto y(t)$:

$$y(t) = Ce^{At} x_0.$$

Observability Gramian:

$$Q := \int_0^{+\infty} (Ce^{At})^T Ce^{At} dt.$$

Gramians

The controllability and observability Gramians satisfy

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

Energy functionals for linear systems

For linear systems, the **energy functionals** are given by

$$L_c(x_0) = x_0^T P^{-1} x_0, \quad L_o(x_0) = x_0^T Q x_0.$$

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- Truncation of states related to small singular values

$$\|y(t) - \hat{y}(t)\|_{L_2} \leq 2 \left(\sum_{j=k+1}^{n} \sigma_j \right) \|u\|_{L_2}.$$

- Generally, exact energy functionals are given by the solutions of **nonlinear Hamilton-Jacobi equations** and **nonlinear Lyapunov**-type equations. [Scherpen '93]

Balanced Truncation

–Main Challenge and a Possible Way–

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- We aim at determining the **algebraic Gramians** for polynomial systems, which
  - provide **bounds for energy functionals** of polynomial control (PC) systems, and
  - allow us to **find the states** that are hard to reach and observe in an efficient way.

- Extending the Volterra series concept QB systems, we propose **the controllability Gramian**.

  [B./Goyal '17]

- Second step, we define an adjoint system of the polynomial system. [Fujimoto et al. '02]
- Based on it, we define the **observability Gramian**.

Theorem [B./Goyal/Pontes '19]

The **reachability Gramian** $(\mathbf{P})$ of a polynomial system solves the **polynomial Lyapunov** equation

$$AP + PA^T + BB^T + \sum_{\xi=2}^{d} H_\xi P^{\textcircled{\xi}} H_\xi^T + \sum_{\eta=1}^{d} N_\eta P^{\textcircled{\eta}} \left(N_\eta\right)^T = 0.$$

The **observability Gramian** $(\mathbf{Q})$ of a polynomial system solves the **polynomial Lyapunov** equation

$$A^T Q + QA + C^T C + \sum_{\xi=1}^{d-1} H_{\xi+1}^{(2)} \left(P^{\textcircled{\xi}} \otimes Q\right) \left(H_{\xi+1}^{(2)}\right)^T + \sum_{\eta=0}^{d-1} N_{\eta+1} \left(P^{\textcircled{\eta}} \otimes Q\right) \left(N_{\eta+1}\right)^T = 0.$$

- We show **bounds** for the **energy functionals** (at least in the neighborhood of the origin), similar to the bilinear and quadratic-bilinear case, as:

$$L_c(x_0) \geq \frac{1}{2} x_0^T P^{-1} x_0, \qquad L_o(x_0) \leq \frac{1}{2} x_0^T Q x_0.$$

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Another interpretation of Gramians in terms of energy functionals [B./Goyal/Pontes '19]

1. Assuming zero initial condition, $\underline{x(t,0,u) \in \text{Im}P}$, $\forall\ t \geq 0$ and all input functions.
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2. If $P > 0$ and the initial state $\in \operatorname{Ker} Q$, then it is unobservable.

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- We propose truncated Gramians that only involve a finite number of kernels.

Definition [B./Goyal/Pontes '19]

The truncated **reachability Gramian** $(\mathbf{P}_\mathcal{T})$ of a polynomial system solves the **linear Lyapunov** equation

$$AP_\mathcal{T} + P_\mathcal{T}A^T + BB^T + \sum_{\xi=2}^{d} H_\xi P_l^{\textcircled{\xi}} H_\xi^T + \sum_{\eta=1}^{d} N_\eta P_l^{\textcircled{\eta}} (N_\eta)^T = 0.$$

The truncated **observability Gramian** $(\mathbf{Q}_\mathcal{T})$ of a polynomial system solves the **linear Lyapunov** equation

$$A^TQ_\mathcal{T} + Q_\mathcal{T}A + C^TC + \sum_{\xi=1}^{d-1} H_{\xi+1}^{(2)} \left(P_l^{\textcircled{\xi}} \otimes Q_l\right) \left(H_{\xi+1}^{(2)}\right)^T + \sum_{\eta=0}^{d-1} N_{\eta+1}^{(2)} \left(P_l^{\textcircled{\eta}} \otimes Q_l\right) \left(N_{\eta+1}^{(2)}\right)^T = 0,$$

where $AP_l + P_lA^T + BB^T = 0$ and $A^TQ_l + Q_lA + C^TC = 0$.

- **Advantage:** Only need to solve four (linear) Lyapunov equations.

Balanced Truncation
–Balancing Algorithm–

- Similar to linear and bilinear cases, **balancing** allows us to find hard to control and observe states, see, e.g., [Antoulas '05, B./Damm '08].

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Provide system matrices $A, H_\xi, N_\eta^k, B, C,$ and order of the reduced system r (optional).

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Low-rank factors: $P \approx SS^T$ and $Q \approx RR^T$.

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Low-rank factors: $P_{\mathcal{T}} \approx SS^T$ and $Q_{\mathcal{T}} \approx RR^T$.

$$AP_l + P_lA^T + BB^T = 0,$$
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$P_l \approx$ , $Q_l \approx$ 

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$$S^T R = U\Sigma V^T = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T, \quad \Sigma_1 \in \mathbb{R}^{r\times r},$$

$$\mathbf{V} = SU_1\Sigma_1^{-\frac{1}{2}}, \mathbf{W} = RV_1\Sigma_1^{-\frac{1}{2}}.$$

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$$\hat{A} = \mathbf{W}^T A \mathbf{V}, \qquad \hat{H}_\xi = \mathbf{W}^T H_\xi \mathbf{V}^{\textcircled{\xi}}, \qquad \hat{N}_\eta^k = \mathbf{W}^T N_\eta^k \mathbf{V}^{\textcircled{\eta}},$$

$$\hat{B} = \mathbf{W}^T B, \qquad \hat{C} = C\mathbf{V}.$$

Governing equations

$$
\begin{aligned}
\epsilon v_t(x,t) &= \epsilon^2 v_{xx}(x,t) + v(1-v)(v-0.1) - w(x,t) + q,\\
w_t(x,t) &= hv(x,t) - \gamma w(x,t) + q,\\
v(x,0) &= 0, \qquad w(x,0) = 0, \qquad x \in [0,L],\\
v_x(0,t) &= u(t), \qquad v_x(L,t) = 0, \qquad t \geq 0,
\end{aligned}
$$

where $\epsilon = 0.015,\ h = 0.5,\ \gamma = 2,\ q = 0.05,\ L = 0.3.$

- After discretization, we obtain a polynomial control (PC) system with cubic nonlinearity of order $n_{pc} = 600$.

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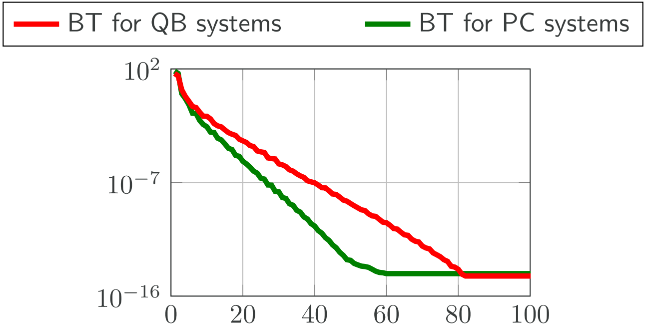
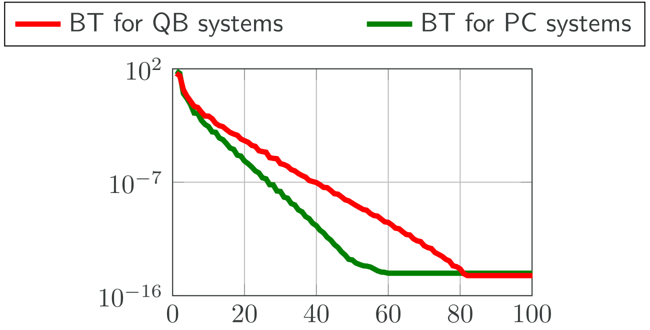
- The outputs of interest $v(0,t), w(0,t)$ are the responses at the left boundary at $x = 0$.
- We compare balanced truncation for PC and QB.



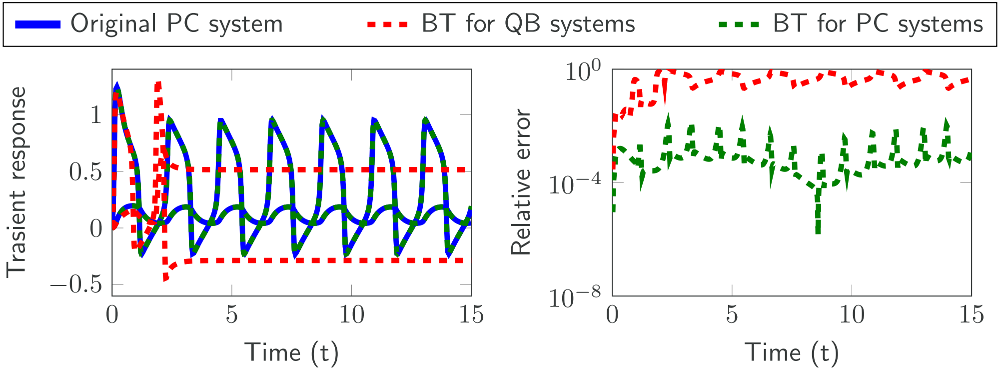
BT for QB systems
BT for PC systems
10^{2}
10^{-7}
10^{-16}
0
20
40
60
80
100



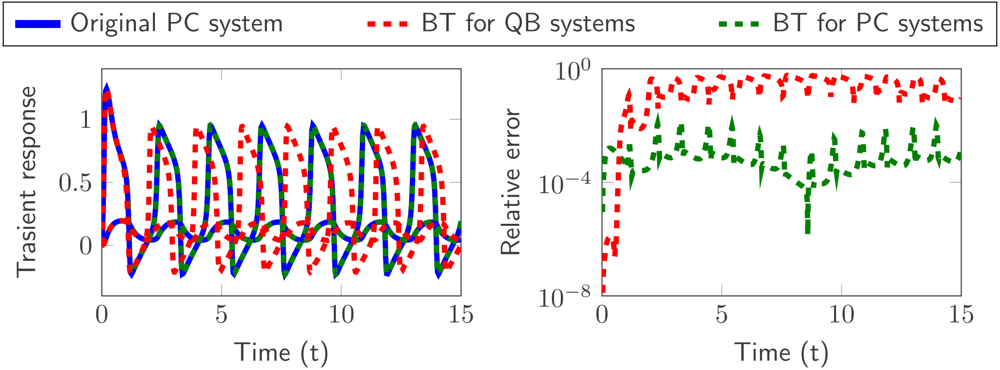
- Decay of singular values for PC systems is faster $\Rightarrow$ smaller reduced order model!

Balanced Truncation

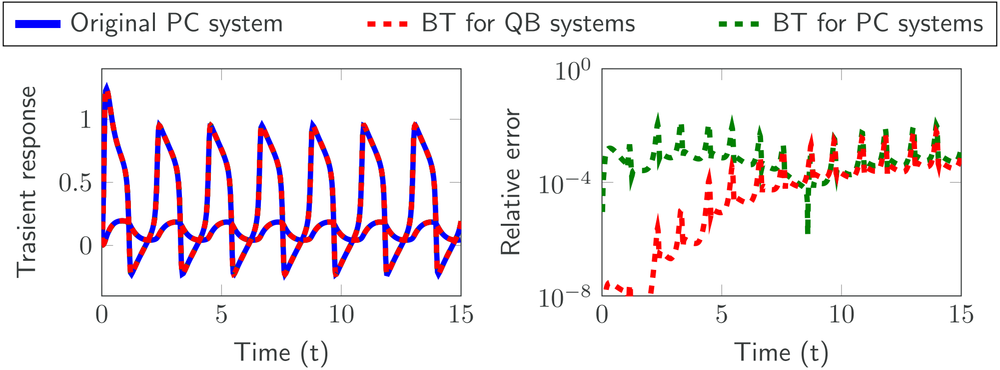
–Time-domain simulations–



- Original PC system of order 600. Original QB of order 900.
- Reduced PC system of order 10. Reduced QB system of order 10.



- Original PC system of order 600. Original QB of order 900.
- Reduced PC system of order 10. Reduced QB system of order 30.



- Original PC system of order 600. Original QB of order 900.
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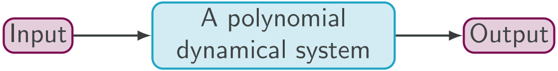
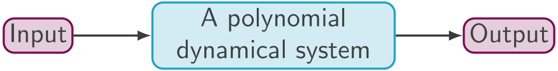
Interpolation-Based Method

–Input-output mapping–





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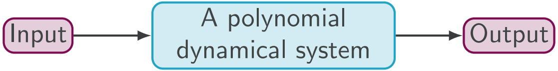

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- Instead of having a single transfer function, we have a sequence of transfer functions.

Interpolation-Based Method

–Input-output mapping–

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + \sum_{\xi=2}^{d} H_\xi x^{\textcircled{\xi}}(t) + \sum_{\eta=2}^{d} N_\xi \left(u(t) \otimes x^{\textcircled{\eta}}(t)(t)\right) + Bu(t),\\
y(t) &= Cx(t), \quad x(0).
\end{aligned}
$$

- As for linear systems, we can define the input-output mapping by generalized transfer functions.
- Instead of having a single transfer function, we have a sequence of transfer functions.
- **Generalized transfer functions** [B./Goyal '19]

$$
\begin{aligned}
\mathbf{F}_L(s_1) &:= C\Phi(s_1)B,\\
\mathbf{F}_H^{(\xi)}(s_1,\dots,s_{\xi+1}) &:= C\Phi(s_{\xi+1})H_\xi\left(\Phi(s_\xi)B \otimes \cdots \otimes \Phi(s_1B)\right),\\
\mathbf{F}_N^{(\eta)}(s_1,\dots,s_{\eta+1}) &:= C\Phi(s_{\eta+1})N_\eta\left(I_m \otimes \Phi(s_\eta)B \otimes \cdots \otimes \Phi(s_1)B\right),
\end{aligned}
$$

where $\Phi(s) := (sE - A)^{-1}$.

Interpolating Reduced System

Goal [B./Goyal '19]

Construct projection matrices $\mathbf{V}$ and $\mathbf{W}$ such that

$$(\mathsf{GTF})^{\sigma}_{\text{original}} = (\mathsf{GTF})^{\sigma}_{\text{reduced}},$$

GTF = Generalized transfer functions

and reduced matrices are constructed via Petro-Galerkin projection:

$$\begin{aligned} \hat{E} &= \mathbf{W}^T A \mathbf{V}, & \hat{A} &= \mathbf{W}^T A \mathbf{V}, & \hat{H}_\xi &= \mathbf{W}^T H_\xi \mathbf{V}^{\textcircled{\xi}}, & \xi &\in \{2, \ldots, d\}, \\ \hat{B} &= \mathbf{W}^T B, & \hat{C} &= C\mathbf{V}, & \hat{N}_\eta &= \mathbf{W}^T H_\eta \mathbf{V}^{\textcircled{\eta}}, & \eta &\in \{1, \ldots, d\}. \end{aligned}$$

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- Extended ideas from linear systems to polynomial systems.

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- Interpolating points play an important role.

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- Extended ideas from linear systems to polynomial systems.
- Interpolating points play an important role.
- To make the **process fully automatic**, we propose a Loewner-type approach to construct good reduced-order systems.

Algorithm to Construct ROMs

Algorithm: Loewner-inspired method for determining reduced-order systems

1. Take $\sigma_i, \mu_i,\ i = 1, \ldots, \mathcal{N}$.

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3. Compute $\mathcal{O} = \begin{cases} \text{range}\left(\Phi(\mu_1)^T C^T, \ldots, \Phi(\mu_{\mathcal{N}})^T C^T\right), \\ \bigcup_{\eta=1}^{d} \bigcup_{i=1}^{\mathcal{N}} \text{range}\left(\Phi(\sigma_i)\left(N_\eta\right)_{(2)} \left(I_m \otimes \Phi(\sigma_i)B \otimes \cdots \otimes \Phi(\sigma_i)B \otimes \Phi(\sigma_i)^T C^T\right)\right) \\ \bigcup_{\xi=2}^{d} \bigcup_{i=1}^{\mathcal{N}} \text{range}\left(\Phi(\sigma_i)\left(H_\xi\right)_{(2)} \left(\Phi(\sigma_i)B \otimes \cdots \otimes \Phi(\sigma_i)B \otimes \Phi(\mu_i)^T C^T\right)\right) \end{cases}.$

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4. Determine Loewner and shifted-Loewner matrices: $\mathbb{L} = -\mathcal{O}^T E \mathcal{R}$, $\quad \mathbb{L}_s = -\mathcal{O}^T A \mathcal{R}$.

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Algorithm to Construct ROMs

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Construct reduced-order system

$\hat{E} = W^T A V,\ \hat{A} = W^T A V,\ \hat{H}_\xi = W^T H_\xi V^{\textcircled{\xi}},$

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Numerical Example

Fitz-Hugh Nagumo model: Governing coupled equation

$$
\begin{aligned}
\epsilon v_t &= \epsilon^2 v_{xx} + v(v-0.1)(1-v) - w + q,\\
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with boundary condition

$$v(x,0)=0, \qquad w(x,0)=0, \qquad x\in(0,L), \qquad v_x(0,t)=i_0(t), \qquad v_x(1,t)=0, \qquad t\geq 0.$$

- To employ the interpolation-based algorithm, we choose 100 interpolation points in a logarithmic way between $\left[10^{-2},10^{2}\right]$ and set $\sigma_i=\mu_i$, $i\in\{1,\ldots,100\}$.

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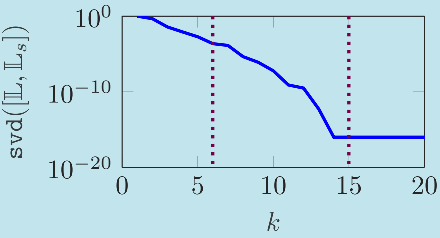
Decay of singular values of $[\mathbb{L}, \mathbb{L}_s]$



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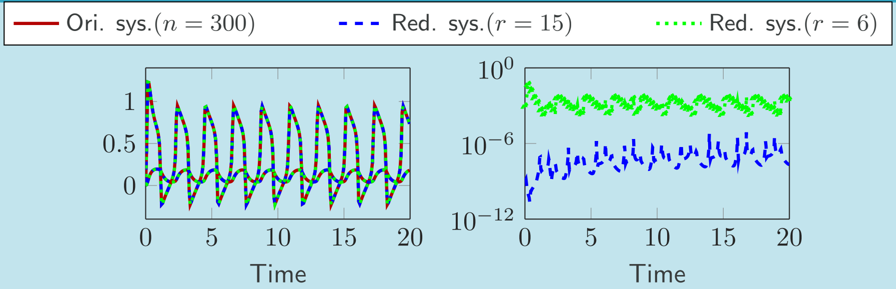
Construction of reduced systems



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Thank you for your attention!!

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