Parametric Model Order Reduction of Dynamical Systems: Survey and Recent Advances

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Overview

1. Introduction to Parametric Model Order Reduction
   - Dynamical Systems
   - Motivating Example: Microsystems/MEMS Design
   - The Parametric Model Order Reduction (PMOR) Problem
   - PMOR $\leftrightarrow$ Multivariate Function Approximation

2. PMOR Methods — a Survey
   - Model Reduction for Linear Parametric Systems
   - Interpolatory Model Reduction
   - PMOR based on Multi-Moment Matching
   - PMOR based on Rational Interpolation
   - Other Approaches

3. PMOR via Bilinearization
   - Parametric Systems as Bilinear Systems
   - $\mathcal{H}_2$-Model Reduction for Bilinear Systems

4. Conclusions and Outlook
Introduction to Parametric Model Order Reduction

Parametric Dynamical Systems

Dynamical Systems

\[ \Sigma(p) : \begin{cases} 
E(p) \dot{x}(t; p) = f(t, x(t; p), u(t), p), & x(t_0) = x_0, \\
y(t; p) = g(t, x(t; p), u(t), p) 
\end{cases} \]

with

- (generalized) states \( x(t; p) \in \mathbb{R}^n \) \((E \in \mathbb{R}^{n \times n})\),
- inputs \( u(t) \in \mathbb{R}^m \),
- outputs \( y(t; p) \in \mathbb{R}^q \), \(b\) is called output equation,
- \( p \in \Omega \subset \mathbb{R}^d \) is a parameter vector, \( \Omega \) is bounded.

Applications:

- Repeated simulation for varying material or geometry parameters, boundary conditions,
- control, optimization and design,
- of models, often generated by FE software (e.g., ANSYS, NASTRAN,...) or automatic tools (e.g., Modelica).
## Introduction to Parametric Model Order Reduction

### Parametric Dynamical Systems

#### Dynamical Systems

\[
\Sigma(p) : \begin{align*}
E(p) \dot{x}(t; p) &= f(t, x(t; p), u(t), p), \quad x(t_0) = x_0, \quad (a) \\
y(t; p) &= g(t, x(t; p), u(t), p) \quad (b)
\end{align*}
\]

with

- (generalized) states \( x(t; p) \in \mathbb{R}^n \) \((E \in \mathbb{R}^{n \times n})\),
- inputs \( u(t) \in \mathbb{R}^m \),
- outputs \( y(t; p) \in \mathbb{R}^q \), \((b)\) is called output equation,
- \( p \in \Omega \subset \mathbb{R}^d \) is a parameter vector, \( \Omega \) is bounded.

PDE and boundary conditions often not accessible!
Introduction to Parametric Model Order Reduction

Linear Parametric Systems

Linear, time-invariant (parametric) systems

\[
E(p) \dot{x}(t; p) = A(p)x(t; p) + B(p)u(t), \quad A(p), E(p) \in \mathbb{R}^{n \times n},
\]
\[
y(t; p) = C(p)x(t; p), \quad B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n}.
\]
Introduction to Parametric Model Order Reduction

Linear Parametric Systems

**Linear, time-invariant (parametric) systems**

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\]

**Laplace Transformation / Frequency Domain**

Application of Laplace transformation \((x(t; p) \mapsto x(s; p), \dot{x}(t; p) \mapsto sx(s; p))\) to linear system with \(x(0; p) \equiv 0:\)

\[
sE(p)x(s; p) = A(p)x(s; p) + B(p)u(s), \quad y(s; p) = C(p)x(s; p),
\]

yields I/O-relation in frequency domain:

\[
y(s; p) = \left( C(p)(sE(p) - A(p))^{-1}B(p) \right)u(s).
\]

\(G(s, p)\) is the parameter-dependent transfer function of \(\Sigma(p)\).
Introduction to Parametric Model Order Reduction

Linear Parametric Systems

**Linear, time-invariant (parametric) systems**

\[
E(p)\dot{x}(t; p) = A(p)x(t; p) + B(p)u(t), \quad A(p), E(p) \in \mathbb{R}^{n \times n},
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y(s; p) = \left( C(p)(sE(p) - A(p))^{-1}B(p) \right)u(s).
\]

\(G(s, p)\) is the parameter-dependent **transfer function** of \(\Sigma(p)\).

**Goal:** **Fast evaluation** of mapping \((u, p) \rightarrow y(s; p)\).
Microgyroscope (butterfly gyro)

- Voltage applied to electrodes induces vibration of wings, resulting rotation due to Coriolis force yields sensor data.
- FE model of second order:
  \[ N = 17.361 \leadsto n = 34.722, \ m = 1, \ q = 12. \]
- Sensor for position control based on acceleration and rotation.

Applications:
- inertial navigation,
- electronic stability control (ESP).

Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model: \( M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t) \).
Introduction to Parametric Model Order Reduction
Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Parametric FE model:

\[ M(d)\ddot{x}(t) + D(\theta, d, \alpha, \beta)\dot{x}(t) + T(d)x(t) = Bu(t), \]

where

\[
\begin{align*}
M(d) &= M_1 + dM_2, \\
D(\theta, d, \alpha, \beta) &= \theta(D_1 + dD_2) + \alpha M(d) + \beta T(d), \\
T(d) &= T_1 + \frac{1}{d} T_2 + dT_3,
\end{align*}
\]

with

- width of bearing: \(d\),
- angular velocity: \(\theta\),
- Rayleigh damping parameters: \(\alpha, \beta\).
Introduction to Parametric Model Order Reduction
Motivating Example: Microsystems/MEMS Design

Microgyroscope (butterfly gyro)

Original... and reduced-order model.
The Parametric Model Order Reduction (PMOR) Problem

**Problem**

Approximate the dynamical system

\[
E(p) \dot{x} = A(p)x + B(p)u, \quad E(p), A(p) \in \mathbb{R}^{n \times n},
\]
\[
y = C(p)x, \quad B(p) \in \mathbb{R}^{n \times m}, C(p) \in \mathbb{R}^{q \times n},
\]

by reduced-order system

\[
\hat{E}(p) \dot{\hat{x}} = \hat{A}(p)\hat{x} + \hat{B}(p)u, \quad \hat{E}(p), \hat{A}(p) \in \mathbb{R}^{r \times r},
\]
\[
\hat{y} = \hat{C}(p)\hat{x}, \quad \hat{B}(p) \in \mathbb{R}^{r \times m}, \hat{C}(p) \in \mathbb{R}^{q \times r},
\]

of order \( r \ll n \), such that

\[
\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\| \cdot \|u\| < \text{tolerance} \cdot \|u\| \quad \forall \ p \in \Omega.
\]
The Parametric Model Order Reduction (PMOR) Problem

Problem

Approximate the dynamical system

\[ E(p) \dot{x} = A(p)x + B(p)u, \quad y = C(p)x, \]

by reduced-order system

\[ \hat{E}(p) \dot{\hat{x}} = \hat{A}(p)\hat{x} + \hat{B}(p)u, \quad \hat{y} = \hat{C}(p)\hat{x}, \]

of order \( r \ll n \), such that

\[ \| y - \hat{y} \| = \| Gu - \hat{G}u \| \leq \| G - \hat{G} \| \cdot \| u \| < \text{tolerance} \cdot \| u \| \quad \forall \ p \in \Omega. \]

\[ \implies \text{Approximation problem: min}_{\text{order } (\hat{G}) \leq r} \| G - \hat{G} \|. \]
Multivariate Function Approximation

- Approximate (for fast evaluation) function $G$, defined on $\mathbb{C} \times \Omega$. 
PMOR ↔ Multivariate Function Approximation

- Approximate (for fast evaluation) function $G$, defined on $\mathbb{C} \times \Omega$.
- But:

  $$G : \mathbb{C} \times \Omega \rightarrow \mathbb{C}^{q \times m}, \quad \Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d],$$
  $$G(s; p_1, \ldots, p_d) \in \mathbb{C}^{q \times m}.$$

  Variables $s$ and $p_j$ have different “meaning” for $G$.
  Dynamical system is in the background!

  Matrix-valued function, require matrix- not entry-wise approximation!
Approximate (for fast evaluation) function $G$, defined on $\mathbb{C} \times \Omega$.

But:

$$G : \mathbb{C} \times \Omega \rightarrow \mathbb{C}^{q \times m}, \quad \Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d],$$

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$\Rightarrow$ Variables $s$ and $p_j$ have different “meaning” for $G$.
Dynamical system is in the background!

$\Rightarrow$ Matrix-valued function, require matrix- not entry-wise approximation!

$G$ is rational in $s$, $n \sim$ degree of denominator polynomial.

$\Rightarrow$ Require approximation to be rational in $s$. 
Approximate (for fast evaluation) function $G$, defined on $\mathbb{C} \times \Omega$.

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- Dynamical system is in the background!
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$G$ is rational in $s$, $n \sim$ degree of denominator polynomial.
- Require approximation to be rational in $s$.

Require structure-preserving approximation, e.g., for control design.
- Need realization as linear parametric system!
PMOR \leftrightarrow \text{Multivariate Function Approximation}

- Approximate (for fast evaluation) function $G$, defined on $\mathbb{C} \times \Omega$.
- But:

  
  \[
  G : \mathbb{C} \times \Omega \rightarrow \mathbb{C}^{q \times m}, \quad \Omega = [\alpha_1, \beta_1] \times \ldots \times [\alpha_d, \beta_d],
  \]

  \[
  G(s; p_1, \ldots, p_d) \in \mathbb{C}^{q \times m}.
  \]

  \[\Rightarrow\] Variables $s$ and $p_j$ have different “meaning” for $G$.
  Dynamical system is in the background!

  \[\Rightarrow\] Matrix-valued function, require matrix- not entry-wise approximation!

- $G$ is rational in $s$, $n \sim$ degree of denominator polynomial.
  \[\Rightarrow\] Require approximation to be rational in $s$.

- Require structure-preserving approximation, e.g., for control design.
  \[\Rightarrow\] Need realization as linear parametric system!

- Also would like to be able to reproduce system dynamics (stability, passivity).
PMOR Methods — a Survey
Model Reduction for Linear Parametric Systems

Parametric System

\[ \Sigma(p) : \begin{cases} 
E(p)\dot{x}(t; p) &= A(p)x(t; p) + B(p)u(t), \\
y(t; p) &= C(p)x(t; p). 
\end{cases} \]
Model Reduction for Linear Parametric Systems

\[ \Sigma(p) : \begin{cases} 
    E(p) \dot{x}(t; p) &= A(p)x(t; p) + B(p)u(t), \\
    y(t; p) &= C(p)x(t; p). 
\end{cases} \]

Appropriate parameter-affine representation:

\[
E(p) = E_0 + e_1(p)E_1 + \ldots + e_{q_E}(p)E_{q_E}, \\
A(p) = A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \\
B(p) = B_0 + b_1(p)B_1 + \ldots + b_{q_B}(p)B_{q_B}, \\
C(p) = C_0 + c_1(p)C_1 + \ldots + c_{q_C}(p)C_{q_C},
\]

allows easy parameter preservation for projection based model reduction.
PMOR Methods — a Survey
Model Reduction for Linear Parametric Systems

Parametric System

\[ \Sigma(p) : \begin{cases} E(p) \dot{x}(t; p) &= A(p)x(t; p) + B(p)u(t), \\ y(t; p) &= C(p)x(t; p). \end{cases} \]

Appropriate parameter-affine representation:

\[ A(p) = A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \ldots \]

allows easy parameter preservation for projection based model reduction.

W.l.o.g. may assume this affine representation:

- Any system can be written in this affine form for some \( q_X \leq n^2 \), but for efficiency, need \( q_X \ll n! \) (\( X \in \{ E, A, B, C \} \))
- Empirical (operator) interpolation yields this structure for ”smooth enough” nonlinearities [Barrault/Maday/Nguyen/Patera 2004].
Parametric System

\[ \Sigma(p) : \begin{cases} 
E(p)\dot{x}(t; p) & = A(p)x(t; p) + B(p)u(t), \\
y(t; p) & = C(p)x(t; p). 
\end{cases} \]

Appropriate parameter-affine representation:

\[ A(p) = A_0 + a_1(p)A_1 + \ldots + a_{q_A}(p)A_{q_A}, \quad \ldots \]

allows easy parameter preservation for projection based model reduction.

Parametric model reduction goal:

preserve parameters as symbolic quantities in reduced-order model:

\[ \hat{\Sigma}(p) : \begin{cases} 
\hat{E}(p)\hat{x}(t; p) & = \hat{A}(p)\hat{x}(t; p) + \hat{B}(p)u(t), \\
\hat{y}(t; p) & = \hat{C}(p)\hat{x}(t; p) 
\end{cases} \]

with states \( \hat{x}(t; p) \in \mathbb{R}^r \) and \( r \ll n \).
Model Reduction for Linear Parametric Systems

Structure-Preservation

Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n \times r}$ with $W^T V = I_r$ ($\sim (VW^T)^2 = VW^T$ is projector), compute

\[
\hat{E}(p) = W^T E_0 V + e_1(p) W^T E_1 V + \ldots + e_{q_E}(p) W^T E_{q_E} V,
\]
\[
= \hat{E}_0 + e_1(p) \hat{E}_1 + \ldots + e_{q_E}(p) \hat{E}_{q_E},
\]

\[
\hat{A}(p) = W^T A_0 V + a_1(p) W^T A_1 V + \ldots + a_{q_A}(p) W^T A_{q_A} V,
\]
\[
= \hat{A}_0 + a_1(p) \hat{A}_1 + \ldots + a_{q_A}(p) \hat{A}_{q_A},
\]

\[
\hat{B}(p) = W^T B_0 + b_1(p) W^T B_1 + \ldots + b_{q_B}(p) W^T B_{q_B},
\]
\[
= \hat{B}_0 + b_1(p) \hat{B}_1 + \ldots + b_{q_B}(p) \hat{B}_{q_B},
\]

\[
\hat{C}(p) = C_0 V + c_1(p) C_1 V + \ldots + c_{q_C}(p) C_{q_C} V,
\]
\[
= \hat{C}_0 + c_1(p) \hat{C}_1 + \ldots + c_{q_C}(p) \hat{C}_{q_C}.
\]
Petrov-Galerkin-type projection

For given projection matrices $V, W \in \mathbb{R}^{n \times r}$ with $W^TV = I_r$ ($\sim (VW^T)^2 = VW^T$ is projector), compute

$$
\hat{E}(p) = W^TE_0V + e_1(p)W^TE_1V + \ldots + e_{q_E}(p)W^TE_{q_E}V,
= \hat{E}_0 + e_1(p)\hat{E}_1 + \ldots + e_{q_E}(p)\hat{E}_{q_E},
$$

$$
\hat{A}(p) = W^TA_0V + a_1(p)W^TA_1V + \ldots + a_{q_A}(p)W^TA_{q_A}V,
= \hat{A}_0 + a_1(p)\hat{A}_1 + \ldots + a_{q_A}(p)\hat{A}_{q_A},
$$

$$
\hat{B}(p) = W^TB_0 + b_1(p)W^TB_1 + \ldots + b_{q_B}(p)W^TB_{q_B},
= \hat{B}_0 + b_1(p)\hat{B}_1 + \ldots + b_{q_B}(p)\hat{B}_{q_B},
$$

$$
\hat{C}(p) = C_0V + c_1(p)C_1V + \ldots + c_{q_C}(p)C_{q_C}V,
= \hat{C}_0 + c_1(p)\hat{C}_1 + \ldots + c_{q_C}(p)\hat{C}_{q_C}.
$$
Computation of reduced-order model by projection

Given a linear (descriptor) system \( E \dot{x} = Ax + Bu, \quad y = Cx \) with transfer function \( G(s) = C(sE - A)^{-1}B \), a reduced-order model is obtained using truncation matrices \( V, W \in \mathbb{R}^{n \times r} \) with \( W^T V = I_r \) (\( \sim (VW^T)^2 = VW^T \) is projector) by computing

\[
\hat{E} = W^T EV, \quad \hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.
\]

Petrov-Galerkin-type (two-sided) projection: \( W \neq V \),

Galerkin-type (one-sided) projection: \( W = V \).
## Computations of reduced-order model by projection

Given a linear (descriptor) system $E\dot{x} = Ax + Bu$, $y = Cx$ with transfer function $G(s) = C(sE - A)^{-1}B$, a reduced-order model is obtained using truncation matrices $V, W \in \mathbb{R}^{n \times r}$ with $W^T V = I_r$ ($\sim (VW^T)^2 = VW^T$ is projector) by computing

$$\hat{E} = W^T E V, \quad \hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$  

**Petrov-Galerkin-type (two-sided) projection:** $W \neq V$,  
**Galerkin-type (one-sided) projection:** $W = V$.

## Rational Interpolation/Moment-Matching

Choose $V, W$ such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$
Theorem (simplified) [Grimme ’97, Villemagne/Skelton ’87]

If

\[
\text{span} \left\{ (s_1 E - A)^{-1}B, \ldots, (s_k E - A)^{-1}B \right\} \subset \text{Ran}(V),
\]
\[
\text{span} \left\{ (s_1 E - A)^{-T}C^T, \ldots, (s_k E - A)^{-T}C^T \right\} \subset \text{Ran}(W),
\]

then

\[
G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.
\]
Theorem (simplified) \([\text{Grimme '97, Villemagne/Skelton '87}]
\]

If

\[
\begin{align*}
\text{span} \{ (s_1 E - A)^{-1} B, \ldots, (s_k E - A)^{-1} B \} & \subset \text{Ran}(V), \\
\text{span} \{ (s_1 E - A)^{-T} C^T, \ldots, (s_k E - A)^{-T} C^T \} & \subset \text{Ran}(W),
\end{align*}
\]

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\]

Remarks:

computation of \(V, W\) from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos \([\text{Grimme '97}]\),
- Iter. Rational Krylov-Alg. (IRKA) \([\text{Antoulas/Beattie/Gugercin '06/’08}]\).
PMOR Methods — a Survey
A Short Introduction to Interpolatory Model Reduction

Theorem (simplified) [Grimme ’97, Villemagne/Skelton ’87]

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\[ \text{span} \left\{ (s_1E - A)^{-T}C^T, \ldots, (s_kE - A)^{-T}C^T \right\} \subset \text{Ran}(W), \]

then

\[ G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k. \]

Remarks:

using Galerkin/one-sided projection \((W \equiv V)\) yields \(G(s_j) = \hat{G}(s_j)\), but in general

\[ \frac{d}{ds} G(s_j) \neq \frac{d}{ds} \hat{G}(s_j). \]
Theorem (simplified) [Grimme ’97, Villemagne/Skelton ’87]

If

\[
\text{span } \{(s_1E - A)^{-1}B, \ldots, (s_kE - A)^{-1}B\} \subset \text{Ran}(V),
\]
\[
\text{span } \{(s_1E - A)^{-T}C^T, \ldots, (s_kE - A)^{-T}C^T\} \subset \text{Ran}(W),
\]

then

\[
G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds} G(s_j) = \frac{d}{ds} \hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.
\]

Remarks:

\(k = 1\), standard Krylov subspace(s) of dimension \(K\):

\[
\text{range } (V) = \mathcal{K}_K((s_1I - A)^{-1}, (s_1I - A)^{-1}B).
\]

\(\leadsto\) moment-matching methods/Padé approximation,

\[
\frac{d^i}{ds^i} G(s_1) = \frac{d^i}{ds^i} \hat{G}(s_1), \quad i = 0, \ldots, K - 1(+K).
\]
**Interpolatory Model Reduction**

**H₂-Model Reduction for Linear Systems**

Consider stable (i.e. \( \Lambda (A) \subset \mathbb{C}^- \)) linear systems \( \Sigma \),

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) =Cx(t) \quad \sim \quad Y(s) = C(sI - A)^{-1}B\ U(s).
\]

Note: \( \mathcal{H}_\infty \)-norm approximation \( \rightsquigarrow \) balanced truncation, Hankel norm approximation.

**System norms**

Two common system norms for measuring approximation quality:

- \( \mathcal{H}_2 \)-norm, \( \| \Sigma \|_{\mathcal{H}_2} = \left( \frac{1}{2\pi} \int_0^{2\pi} \text{tr} \left( (G^T(-j\omega)G(j\omega)) \right) d\omega \right)^{\frac{1}{2}}, \)

- \( \mathcal{H}_\infty \)-norm, \( \| \Sigma \|_{\mathcal{H}_\infty} = \sup_{\omega \in \mathbb{R}} \sigma_{\text{max}} (G(j\omega)), \)

where

\[
G(s) = C \ (sI - A)^{-1} B.
\]
Interpolatory Model Reduction

Error system and $\mathcal{H}_2$-Optimality

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $G(s) - \hat{G}(s)$ which can be realized by

\[
A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.
\]
Interpolatory Model Reduction

Error system and $\mathcal{H}_2$-Optimality

[Meier/Luenberger 1967]

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $G(s) - \hat{G}(s)$ which can be realized by

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A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.
\]

Assuming a coordinate system in which $\hat{A}$ is diagonal and taking derivatives of

\[
\|G(\cdot) - \hat{G}(\cdot)\|_{\mathcal{H}_2}^2
\]

with respect to free parameters in $\Lambda(\hat{A}), \hat{B}, \hat{C} \rightsquigarrow$ first-order necessary $\mathcal{H}_2$-optimality conditions (SISO)

\[
G(-\hat{\lambda}_i) = \hat{G}(-\hat{\lambda}_i), \quad \quad G'(-\hat{\lambda}_i) = \hat{G}'(-\hat{\lambda}_i),
\]

where $\hat{\lambda}_i$ are the poles of the reduced system $\hat{\Sigma}$. 
Interpolatory Model Reduction

Error system and $\mathcal{H}_2$-Optimality

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $G(s) - \hat{G}(s)$ which can be realized by

\[
A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}.
\]

First-order necessary $\mathcal{H}_2$-optimality conditions (MIMO):

\[
G(-\hat{\lambda}_i)\tilde{B}_i = \hat{G}(-\hat{\lambda}_i)\tilde{B}_i, \quad \text{for } i = 1, \ldots, \hat{n},
\]

\[
\tilde{C}_i^T G(-\hat{\lambda}_i) = \tilde{C}_i^T \hat{G}(-\hat{\lambda}_i), \quad \text{for } i = 1, \ldots, \hat{n},
\]

\[
\tilde{C}_i^T H'(-\hat{\lambda}_i)\tilde{B}_i = \tilde{C}_i^T \hat{G}'(-\hat{\lambda}_i)\tilde{B}_i \quad \text{for } i = 1, \ldots, \hat{n},
\]

where $\hat{A} = R\hat{\Lambda}R^{-T}$ is the spectral decomposition of the reduced system and $\tilde{B} = \hat{B}^T R^{-T}$, $\tilde{C} = \hat{C} R$. 
Interpolatory Model Reduction

Error system and $\mathcal{H}_2$-Optimality

[Meier/Luenberger 1967]

In order to find an $\mathcal{H}_2$-optimal reduced system, consider the error system $G(s) - \hat{G}(s)$ which can be realized by

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$$

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$$

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$$

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$$

$$
\iff \quad \text{vec} \left( I_q \right)^T \left( e_j e_i^T \otimes C \right) \left( -\hat{\Lambda} \otimes I_n - I_{\hat{n}} \otimes A \right)^{-1} \left( \tilde{B}^T \otimes B \right) \text{vec} \left( I_m \right)
$$

$$
= \text{vec} \left( I_q \right)^T \left( e_j e_i^T \otimes \hat{C} \right) \left( -\hat{\Lambda} \otimes I_{\hat{n}} - I_{\hat{n}} \otimes \hat{A} \right)^{-1} \left( \tilde{B}^T \otimes \hat{B} \right) \text{vec} \left( I_m \right),
$$

for $i = 1, \ldots, \hat{n}$ and $j = 1, \ldots, q$. 
Interpolatory Model Reduction
Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection \( \mathcal{P} = VW^T \), i.e.

\[
\hat{G}(s) = CV \left( sl - W^T AV \right)^{-1} W^T B,
\]
Interpolatory Model Reduction

Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV \left( sI - W^T AV \right)^{-1} W^T B,$$

where $V$ and $W$ are given as

$$V = \left[ (-\mu_1 I - A)^{-1} B, \ldots, (-\mu_r I - A)^{-1} B \right],$$
$$W = \left[ (-\mu_1 I - A^T)^{-1} C^T, \ldots, (-\mu_r I - A^T)^{-1} C^T \right].$$
Interpolatory Model Reduction

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Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$. 
Interpolatory Model Reduction
Interpolation of the Transfer Function [Grimme 1997]

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV(sI - W^T AV)^{-1} W^T B,$$

where $V$ and $W$ are given as

\begin{align*}
V &= \begin{bmatrix}
(-\mu_1 I - A)^{-1} B, 
& \cdots, 
& (-\mu_r I - A)^{-1} B
\end{bmatrix}, \\
W &= \begin{bmatrix}
(-\mu_1 I - A^T)^{-1} C^T, 
& \cdots, 
& (-\mu_r I - A^T)^{-1} C^T
\end{bmatrix}.
\end{align*}

Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$.

Starting with an initial guess for $\hat{\Lambda}$ and setting $\mu_i \equiv \hat{\lambda}_i \rightsquigarrow$ iterative algorithms (IRKA/MIRIAM) that yield $\mathcal{H}_2$-optimal models.

[GUGERCIN ET AL. 2006/08], [BUNSE-GERSTNER ET AL. 2007],

[ VAN DOOREN ET AL. 2008]
Interpolatory Model Reduction
The Basic IRKA Algorithm

Algorithm 1 IRKA (MIMO version/MIRIAm)

**Input:** $A$ stable, $B$, $C$, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.

**Output:** $A^{opt}$, $B^{opt}$, $C^{opt}$

1: while $(\max_{j=1,...,r} \left\{ \frac{|\mu_j - \mu_j^{old}|}{|\mu_j|} \right\} > \delta)$ do

2: diag \{\mu_1,\ldots,\mu_r\} := $T^{-1} \hat{A} T$ = spectral decomposition, \[\tilde{B} = \hat{B}^H T^{-T}, \tilde{C} = \hat{C} T.\]

3: $V = \left[ (-\mu_1 I - A)^{-1} B \tilde{b}_1, \ldots , (-\mu_r I - A)^{-1} B \tilde{b}_r \right]$  

4: $W = \left[ (-\mu_1 I - A^T)^{-1} C^T \tilde{c}_1, \ldots , (-\mu_r I - A^T)^{-1} C^T \tilde{c}_r \right]$  

5: $V =$ orth($V$), $W =$ orth($W$), $W = W (V^H W)^{-1}$

6: $\hat{A} = W^H A V$, $\hat{B} = W^H B$, $\hat{C} = C V$

7: end while

8: $A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$
PMOR based on Multi-Moment Matching

Idea: choose appropriate frequency parameter $\hat{s}$ and parameter vector $\hat{p}$, expand into multivariate power series about $(\hat{s}, \hat{p})$ and compute reduced-order model, so that

$$G(s, p) = \hat{G}(s, p) + O (|s - \hat{s}|^K + \|p - \hat{p}\|^L + |s - \hat{s}|^k \|p - \hat{p}\|^\ell),$$

i.e., first $K, L, k + \ell$ (mostly: $K = L = k + \ell$) coefficients (multi-moments) of Taylor/Laurent series coincide.
PMOR based on Multi-Moment Matching

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$$G(s, p) = \hat{G}(s, p) + \mathcal{O}(|s - \hat{s}|^K + \|p - \hat{p}\|^L + |s - \hat{s}|^k \|p - \hat{p}\|^\ell),$$

i.e., first $K, L, k + \ell$ (mostly: $K = L = k + \ell$) coefficients (multi-moments) of Taylor/Laurent series coincide.

Algorithms:

- [Daniel et al. 2004]: explicit computation of moments, numerically unstable.
- [Farle et al. 2006/07]: Krylov subspace approach, only polynomial parameter-dependance, numerical properties not clear, but appears to be robust.
- [Feng/B. 2007/14]: Arnoldi-MGS method, employ recursive dependance of multi-moments, numerically robust, $r$ often larger as with [Farle et al.].
PMOR based on Multi-Moment Matching

Numerical Examples

Electro-chemical SEM:
compute cyclic voltammogram based on FEM model

\[ E\dot{x}(t) = (A_0 + p_1 A_1 + p_2 A_2)x(t) + Bu(t), \quad y(t) = c^T x(t), \]

where \( n = 16,912, m = 3, A_1, A_2 \) diagonal.

\[ K = L = k + \ell = 4 \Rightarrow r = 26 \]

\[ K = L = k + \ell = 9 \Rightarrow r = 86 \]
Theorem 1 \cite{Baur/Beattie/B./Gugercin 2007/2011}

Let \( \hat{G}(s, p) := \hat{C}(p)(s\hat{E}(p) - \hat{A}(p))^{-1}\hat{B}(p) \)
\[ = C(p)V(sW^TE(p)V - W^TA(p)V)^{-1}W^TB(p). \]

Suppose \( \hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T \) and \( \hat{s} \in \mathbb{C} \) are chosen such that both \( \hat{s} E(\hat{p}) - A(\hat{p}) \) and \( \hat{s} \hat{E}(\hat{p}) - \hat{A}(\hat{p}) \) are invertible.

If
\[ (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p}) \in \text{Ran}(V) \]
or
\[ \left( C(\hat{p}) (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} \right)^T \in \text{Ran}(W), \]
then \( G(\hat{s}, \hat{p}) = \hat{G}(\hat{s}, \hat{p}) \).
PMOR based on Rational Interpolation

Theory: Interpolation of the Transfer Function


Let

\[ \hat{G}(s, p) := \hat{C}(p)(s\hat{E}(p) - \hat{A}(p))^{-1}\hat{B}(p) \]

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Suppose \( \hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T \) and \( \hat{s} \in \mathbb{C} \) are chosen such that both \( \hat{s} E(\hat{p}) - A(\hat{p}) \) and \( \hat{s} \hat{E}(\hat{p}) - \hat{A}(\hat{p}) \) are invertible.

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then \( G(\hat{s}, \hat{p}) = \hat{G}(\hat{s}, \hat{p}) \).

Note: result extends to MIMO case using tangential interpolation:

Let \( 0 \neq b \in \mathbb{R}^m, 0 \neq c \in \mathbb{R}^q \) be arbitrary.

a) If \( (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p})b \in \text{Ran}(V) \), then \( G(\hat{s}, \hat{p})b = \hat{G}(\hat{s}, \hat{p})b \);

b) If \( \left( c^T C(\hat{p}) (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} \right)^T \in \text{Ran}(W) \), then \( c^T G(\hat{s}, \hat{p}) = c^T \hat{G}(\hat{s}, \hat{p}) \).
Theorem 2 [Baur/Beattie/B./Gugercin ’07/’09]

Suppose that $E(p), A(p), B(p), C(p)$ are $C^1$ in a neighborhood of $\hat{p} = [\hat{p}_1, ..., \hat{p}_d]^T$ and that both $\hat{s} E(\hat{p}) - A(\hat{p})$ and $\hat{s} \hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible. If

$$\quad (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p}) \in \text{Ran}(V)$$

and

$$\quad \left( C(\hat{p}) (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} \right)^T \in \text{Ran}(W),$$

then

$$\nabla_p G(\hat{s}, \hat{p}) = \nabla_p G_r(\hat{s}, \hat{p}), \quad \frac{\partial}{\partial s} G(\hat{s}, \hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s}, \hat{p}).$$
PMOR based on Rational Interpolation

Theory: Interpolation of the Parameter Gradient

Theorem 2 [Baur/Beattie/B./Gugercin ’07/’09]

Suppose that $E(p), A(p), B(p), C(p)$ are $C^1$ in a neighborhood of
$\hat{p} = [\hat{p}_1, ..., \hat{p}_d]^T$ and that both $\hat{s} E(\hat{p}) - A(\hat{p})$ and $\hat{s} \hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are
invertible. If

$$(\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p}) \in \text{Ran}(V)$$

and

$$\left( C(\hat{p}) (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} \right)^T \in \text{Ran}(W),$$

then

$$\nabla_p G(\hat{s}, \hat{p}) = \nabla_p G_r(\hat{s}, \hat{p}), \quad \frac{\partial}{\partial s} G(\hat{s}, \hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s}, \hat{p}).$$

Note: result extends to MIMO case using tangential interpolation:

Let $0 \neq b \in \mathbb{R}^m$, $0 \neq c \in \mathbb{R}^q$ be arbitrary. If

$$(\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p}) b \in \text{Ran}(V)$$

and

$$\left( c^T C(\hat{p}) (\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} \right)^T \in \text{Ran}(W),$$

then

$$\nabla_p c^T G(\hat{s}, \hat{p}) b = \nabla_p c^T \hat{G}(\hat{s}, \hat{p}) b.$$
PMOR based on Rational Interpolation

Theory: Interpolation of the Parameter Gradient

Theorem 2 [Baur/Beattie/B./Gugercin '07/'09]

Suppose that $E(p), A(p), B(p), C(p)$ are $C^1$ in a neighborhood of $\hat{p} = [\hat{p}_1, \ldots, \hat{p}_d]^T$ and that both $\hat{s} E(\hat{p}) - A(\hat{p})$ and $\hat{s} \hat{E}(\hat{p}) - \hat{A}(\hat{p})$ are invertible. If

$$(\hat{s} E(\hat{p}) - A(\hat{p}))^{-1} B(\hat{p}) \in \text{Ran}(V)$$

and

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then

$$\nabla_p G(\hat{s}, \hat{p}) = \nabla_p G_r(\hat{s}, \hat{p}), \quad \frac{\partial}{\partial s} G(\hat{s}, \hat{p}) = \frac{\partial}{\partial s} \hat{G}(\hat{s}, \hat{p}).$$

1. Assertion of theorem satisfies necessary conditions for surrogate models in trust region methods [Alexandrov/Dennis/Lewis/Torczon ’98].

2. Approximation of gradient allows use of reduced-order model for sensitivity analysis.
PMOR based on Rational Interpolation

Algorithm

Generic implementation of interpolatory PMOR

Define $\mathcal{A}(s, p) := sE(p) - A(p)$.

1. Select “frequencies” $s_1, \ldots, s_k \in \mathbb{C}$ and parameter vectors $p^{(1)}, \ldots, p^{(\ell)} \in \mathbb{R}^d$.

2. Compute (orthonormal) basis of

$$\mathcal{V} = \text{span} \left\{ \mathcal{A}(s_1, p^{(1)})^{-1} B(p^{(1)}), \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-1} B(p^{(\ell)}) \right\}.$$  

3. Compute (orthonormal) basis of

$$\mathcal{W} = \text{span} \left\{ \mathcal{A}(s_1, p^{(1)})^{-T} C(p^{(1)})^{T}, \ldots, \mathcal{A}(s_k, p^{(\ell)})^{-T} C(p^{(\ell)})^{T} \right\}.$$  

4. Set $\mathcal{V} := [v_1, \ldots, v_{k\ell}]$, $\tilde{\mathcal{W}} := [w_1, \ldots, w_{k\ell}]$, and $\mathcal{W} := \tilde{\mathcal{W}}(\tilde{\mathcal{W}}^T \mathcal{V})^{-1}$. (Note: $r = k\ell$).

5. Compute

$$\begin{align*}
\hat{A}(p) & := \mathcal{W}^T A(p) \mathcal{V}, \\
\hat{B}(p) & := \mathcal{W}^T B(p) \mathcal{V}, \\
\hat{C}(p) & := \mathcal{W}^T C(p) \mathcal{V}, \\
\hat{E}(p) & := \mathcal{W}^T E(p) \mathcal{V}.
\end{align*}$$
PMOR based on Rational Interpolation

Remarks

- If directional derivatives w.r.t. $p$ are included in $\text{Ran}(V)$, $\text{Ran}(W)$, then also the Hessian of $G(\hat{s}, \hat{p})$ is interpolated by the Hessian of $\hat{G}(\hat{s}, \hat{p})$. 

Optimal choice of interpolation frequencies $s_k$ and parameter vectors $p(k)$ in general is an open problem. For prescribed parameter vectors $p(k)$, we can use corresponding $H_2$-optimal frequencies $s_k, \ell = 1, \ldots, r_k$ computed by IRKA, i.e., reduced-order systems $\hat{G}(k)^*$ so that 

$$
\|G(\cdot, p(k)) - \hat{G}(k)^*(\cdot)\|_{H_2} = \min \text{order}(\hat{G}) = r_k \hat{G} \text{ stable} 
$$

where 

$$
\|G(\cdot, p(k)) - \hat{G}(k)^*(\cdot)\|_{H_2} = \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \|G(\omega)\|^2 F d\omega \right)^{1/2}.
$$

Optimal choice of interpolation frequencies $s_k$ and parameter vectors $p(k)$ possible for special cases.
PMOR based on Rational Interpolation

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- If directional derivatives w.r.t. $p$ are included in $\text{Ran}(V)$, $\text{Ran}(W)$, then also the Hessian of $G(\hat{s}, \hat{p})$ is interpolated by the Hessian of $\hat{G}(\hat{s}, \hat{p})$.
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PMOR based on Rational Interpolation

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$$
\| G(\cdot, p^{(k)}) - \hat{G}^{(k)}(\cdot) \|_{\mathcal{H}_2} = \min_{\text{order}(\hat{G})=r_k} \| G(\cdot, p^{(k)}) - \hat{G}^{(k)}(\cdot) \|_{\mathcal{H}_2},
$$

where

$$
\| G \|_{\mathcal{H}_2} := \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \| G(j\omega) \|_F^2 d\omega \right)^{1/2}.
$$
PMOR based on Rational Interpolation

Remarks

- If directional derivatives w.r.t. $p$ are included in $\text{Ran}(V)$, $\text{Ran}(W)$, then also the Hessian of $G(\hat{s}, \hat{p})$ is interpolated by the Hessian of $\hat{G}(\hat{s}, \hat{p})$.

- Choice of optimal interpolation frequencies $s_k$ and parameter vectors $p^{(k)}$ in general is an open problem.

- For prescribed parameter vectors $p^{(k)}$, we can use corresponding $\mathcal{H}_2$-optimal frequencies $s_{k,\ell}$, $\ell = 1, \ldots, r_k$ computed by IRKA, i.e., reduced-order systems $\hat{G}^{(k)}_*$ so that

$$\| G(., p^{(k)}) - \hat{G}^{(k)}_*(.) \|_{\mathcal{H}_2} = \min_{\text{order}(\hat{G})=r_k, \hat{G} \text{ stable}} \| G(., p^{(k)}) - \hat{G}^{(k)}(.) \|_{\mathcal{H}_2},$$

where

$$\| G \|_{\mathcal{H}_2} := \left( \frac{1}{2\pi} \int_{-\infty}^{+\infty} \| G(j\omega) \|_F^2 \, d\omega \right)^{1/2}.$$ 

- Optimal choice of interpolation frequencies $s_k$ and parameter vectors $p^{(k)}$ possible for special cases.
PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

- Important requirement for a compact model of thermal conduction is boundary condition independence.
- The thermal problem is modeled by the heat equation, where heat exchange through device interfaces is modeled by convection boundary conditions containing film coefficients \( \{p_i\}_{i=1}^3 \), to describe the heat exchange at the \( i \)th interface.
- Spatial semi-discretization leads to

\[
E \dot{x}(t) = (A_0 + \sum_{i=1}^3 p_i A_i)x(t) + bu(t), \quad y(t) = c^Tx(t),
\]

where \( n = 4, 257, A_i, i = 1, 2, 3 \), are diagonal.

PMOR based on Rational Interpolation

Numerical Example: Thermal Conduction in a Semiconductor Chip

Choose 2 interpolation points for parameters ("important" configurations), 8/7 interpolation frequencies are picked $H_2$ optimal by IRKA. $\Rightarrow k = 2, \ell = 8, 7$, hence $r = 15$.

$p_3 = 1, p_1, p_2 \in [1, 10^4]$. 

![Relative $H_\infty$ error for $p_3 = 1$](image)
Other Approaches

- Transfer function interpolation (= output interpolation in frequency domain)  
  
  [B./Baur 2008]
PMOR Methods — a Survey

Other Approaches

- Transfer function interpolation (= output interpolation in frequency domain)  
  [B./Baur 2008]
- Matrix interpolation  
  [Panzer/Mohring/Eid/Lohmann 2010, Amsallam/Farhat 2011]
Other Approaches

- Transfer function interpolation (≡ output interpolation in frequency domain) [Baur 2008]
- Matrix interpolation [Panzer/Mohring/Eid/Lohmann 2010, Amsallam/Farhat 2011]
- Manifold interpolation [Amsallam/Farhat/... 2008]
Other Approaches

- Transfer function interpolation (= output interpolation in frequency domain) \[ \text{[B./Baur 2008]} \]
- Matrix interpolation \[ \text{[Panzer/Mohring/Eid/Lohmann 2010, Amsallam/Farhat 2011]} \]
- Manifold interpolation \[ \text{[Amsallam/Farhat/... 2008]} \]
- Proper orthogonal/generalized decomposition (POD/PGD) \[ \text{[Kunisch/Volkwein, Hinze, Willcox, Nouy, ...]} \]
PMOR Methods — a Survey

Other Approaches

- Transfer function interpolation (= output interpolation in frequency domain)  
  [B./Baur 2008]

- Matrix interpolation  
  [Panzer/Mohring/Eid/Lohmann 2010, Amsallam/Farhat 2011]

- Manifold interpolation  
  [Amsallam/Farhat/... 2008]

- Proper orthogonal/generalized decomposition (POD/PGD)  
  [Kunisch/Volkwein, Hinze, Willcox, Nouy, ...]

- Reduced basis method (RBM)  
  [Haasdonk, Maday, Patera, Prud’homme, Rozza, Urban, ...]
Consider bilinear control systems:

\[
\Sigma : \begin{cases} 
\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\
y(t) = Cx(t), \quad x(0) = x_0,
\end{cases}
\]

where \( A, A_i \in \mathbb{R}^{n\times n}, \; B \in \mathbb{R}^{n\times m}, \; C \in \mathbb{R}^{q\times n}. \)
Parametric Systems as Bilinear Systems

Linear Parametric Systems — An Alternative Interpretation

Consider bilinear control systems:

\[
\Sigma : \begin{cases}
\dot{x}(t) = Ax(t) + \sum_{i=1}^{m} A_i x(t) u_i(t) + Bu(t), \\
y(t) = Cx(t), \quad x(0) = x_0,
\end{cases}
\]

where \( A, A_i \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{q \times n} \).

Key Observation

Consider parameters as additional inputs, a linear parametric system

\[
\dot{x}(t) = Ax(t) + \sum_{i=1}^{m_p} a_i(p) A_i x(t) + B_0 u_0(t), \quad y(t) = Cx(t)
\]

with \( B_0 \in \mathbb{R}^{n \times m_0} \) can be interpreted as bilinear system:

\[
\begin{align*}
    u(t) &:= [a_1(p) \ldots a_{m_p}(p) u_0(t)]^T, \\
    B &:= [0 \ldots 0 B_0] \in \mathbb{R}^{n \times m}, \quad m = m_p + m_0.
\end{align*}
\]
Linear parametric systems can be interpreted as bilinear systems.
Linear parametric systems can be interpreted as bilinear systems.

**Consequence**

Model order reduction techniques for bilinear systems can be applied to linear parametric systems!

**Here:**

- Balanced truncation,
- $\mathcal{H}_2$ optimal model reduction.
$H_2$-Model Reduction for Bilinear Systems

Some background

Consider bilinear system ($m = 1$, i.e. SISO)

$$\Sigma : \{ \dot{x}(t) = Ax(t) + A_1x(t)u(t) + Bu(t), \quad y(t) = Cx(t). \}$$
\( \mathcal{H}_2 \)-Model Reduction for Bilinear Systems

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Output Characterization (SISO): Volterra series

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y(t) = \sum_{k=1}^{\infty} \int_{0}^{t} \cdots \int_{0}^{t_{k-1}} K(t_1, \ldots, t_k) u(t-t_1-\cdots-t_k) \cdots u(t-t_k) dt_k \cdots dt_1,
\]

with kernels \( K(t_1, \ldots, t_k) = Ce^{At_k} A_1 \cdots e^{At_2} A_1 e^{At_1} B \).
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G_k(s_1, \ldots, s_k) = C(s_k I - A)^{-1} A_1 \cdots (s_2 I - A)^{-1} A_1(s_1 I - A)^{-1} B.
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**H₂-Model Reduction for Bilinear Systems**

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**Bilinear H₂-norm:**

\[
\|\Sigma\|_{\mathcal{H}_2} := \left( \text{tr} \left( \left( \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{1}{(2\pi)^k} G_k(i\omega_1, \ldots, i\omega_k) G_k^T(i\omega_1, \ldots, i\omega_k) \right) \right) \right)^{1/2}.
\]

[Zhang/Lam 2002]
\( \mathcal{H}_2 \)-Model Reduction for Bilinear Systems

Measuring the Approximation Error

**Lemma** [B./Breiten 2012]

Let \( \Sigma \) denote a bilinear system. Then, the \( \mathcal{H}_2 \)-norm is given as:

\[
\| \Sigma \|_{\mathcal{H}_2}^2 = (\text{vec}(I_q))^T (C \otimes C) \left( -A \otimes I - I \otimes A - \sum_{i=1}^{m} A_i \otimes A_i \right)^{-1} (B \otimes B) \text{vec}(I_m).
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$\mathcal{H}_2$-Model Reduction for Bilinear Systems

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\textbf{Error System}

In order to find an $\mathcal{H}_2$-optimal reduced system, define the error system $\Sigma^{err} := \Sigma - \hat{\Sigma}$ as follows:

$$A^{err} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}, \quad A_i^{err} = \begin{bmatrix} A_i & 0 \\ 0 & \hat{A}_i \end{bmatrix}, \quad B^{err} = \begin{bmatrix} B \\ \hat{B} \end{bmatrix}, \quad C^{err} = \begin{bmatrix} C & -\hat{C} \end{bmatrix}. $$
\( H_2 \)-Model Reduction

\( H_2 \)-Optimality Conditions

Assume \( \hat{\Sigma} \) is given in coordinate system induced by eigenvalue decomposition of \( \hat{A} \):

\[
\hat{A} = R \Lambda R^{-1}, \quad \tilde{A}_i = R^{-1} \hat{A}_i R, \quad \tilde{B} = R^{-1} \hat{B}, \quad \tilde{C} = \hat{C} R.
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Using \( \Lambda, \tilde{A}_i, \tilde{B}, \tilde{C} \) as optimization parameters, we can derive necessary conditions for \( H_2 \)-optimality, e.g.
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\[
(vec(I_q))^T \left( e_j e^T_\ell \otimes C \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{i=1}^{m} \tilde{A}_i \otimes A_i \right)^{-1} \left( \tilde{B} \otimes \hat{B} \right) vec(I_m) = (vec(I_q))^T \left( e_j e^T_\ell \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes \hat{A} - \sum_{i=1}^{m} \tilde{A}_i \otimes \hat{A}_i \right)^{-1} \left( \tilde{B} \otimes \hat{B} \right) vec(I_m).
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\begin{align*}
(\text{vec}(l_q))^T \left( e_j e^T_\ell \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_\hat{n} \otimes \hat{A} - \sum_{i=1}^m \tilde{A}_i \otimes \tilde{A}_i \right)^{-1} \left( \tilde{B} \otimes \tilde{B} \right) \text{vec}(l_m) \\
= (\text{vec}(l_q))^T \left( e_j e^T_\ell \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_\hat{n} \otimes \hat{A} - \sum_{i=1}^m \tilde{A}_i \otimes \tilde{A}_i \right)^{-1} \left( \tilde{B} \otimes \tilde{B} \right) \text{vec}(l_m).
\end{align*}
\]

Connection to interpolation of transfer functions?
**$\mathcal{H}_2$-Model Reduction**

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$$\begin{align*}
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= (\text{vec}(I_q))^T \left( e_j e_\ell^T \otimes \hat{C} \right) \left( -\Lambda \otimes I_n - I_\hat{n} \otimes \hat{A} - \sum_{i=1}^{m} \tilde{A}_i \otimes \hat{A}_i \right)^{-1} \left( \tilde{B} \otimes \hat{B} \right) \text{vec}(I_m).
\end{align*}$$

For $A_i \equiv 0$, this is equivalent to

$$G(-\lambda_\ell) \tilde{B}_\ell^T = \hat{G}(-\lambda_\ell) \tilde{B}_\ell^T$$

$\rightsquigarrow$ tangential interpolation at mirror images of reduced system poles!
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$\leadsto$ tangential interpolation at mirror images of reduced system poles!

**Note:** [FLAGG 2011] shows equivalence to interpolating the Volterra series!
A First Iterative Approach

Algorithm 2 Bilinear IRKA

**Input:** \( A, A_i, B, C, \hat{A}, \hat{A}_i, \hat{B}, \hat{C} \)

**Output:** \( A^{opt}, A_i^{opt}, B^{opt}, C^{opt} \)

1. **while** (change in \( \Lambda > \epsilon \)) **do**
2. \( R^{-1} = \hat{A}, \tilde{B} = R^{-1} \hat{B}, \tilde{C} = \hat{C} R, \tilde{A}_i = R^{-1} \hat{A}_i R \)
3. \( \text{vec}(V) = \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes A - \sum_{i=1}^{m} \tilde{A}_i \otimes A_i \right)^{-1} \left( \tilde{B} \otimes B \right) \text{vec}(I_m) \)
4. \( \text{vec}(W) = \left( -\Lambda \otimes I_n - I_{\hat{n}} \otimes A^T - \sum_{i=1}^{m} \tilde{A}_i^T \otimes A_i^T \right)^{-1} \left( \tilde{C}^T \otimes C^T \right) \text{vec}(I_q) \)
5. \( V = \text{orth}(V), W = \text{orth}(W) \)
6. \( \hat{A} = (W^T V)^{-1} W^T A V, \hat{A}_i = (W^T V)^{-1} W^T A_i V, \)
   \( \hat{B} = (W^T V)^{-1} W^T B, \hat{C} = C V \)
7. **end while**
8. \( A^{opt} = \hat{A}, A_i^{opt} = \hat{A}_i, B^{opt} = \hat{B}, C^{opt} = \hat{C} \)
H₂-Model Reduction for Bilinear Systems

Industrial Case Study: Thermal Analysis of Electrical Motor

- Thermal simulations to detect whether temperature changes lead to fatigue or deterioration of employed materials.
- Main heat source: thermal losses resulting from current stator coil/rotor.
- Many different current profiles need to be considered to predict whether temperature on certain parts of the motor remains in feasible region.
- Finite element analysis on rather complicated geometries \( \mapsto \) large-scale linear models with many (here: 7/13) parameters.

Schematic view of an electrical motor.

Bosch integrated motor generator used in hybrid variants of Porsche Cayenne, VW Touareg.

Pictures: © BOSCH
**H₂-Model Reduction for Bilinear Systems**

**Industrial Case Study: Thermal Analysis of Electrical Motor**

- FEM analysis of thermal model $\rightarrow$ linear parametric systems with $n = 41, 199$, $m = 4$ inputs, and $d = 13$ parameters,
- measurements taken at $q = 4$ heat sensors;
- time for 1 transient simulation in COMSOL\textsuperscript{®} $\sim 90$min;
- ROM order $\hat{n} = 300$, time for 1 transient simulation $\sim 15$sec.
- Legend: Temperature curves for six different values (5, 25, 45, 65, 85, 100$[W/m^2K]$) of the heat transfer coefficient on the coil.
Conclusions and Outlook

- We have reviewed the most popular PMOR methods developed in the last decade, in particular those based on rational interpolation.

  Open problem in general: optimal interpolation points.
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- **Balanced truncation:**
  - Under certain assumptions, we can expect the existence of low-rank approximations to the solution of generalized Lyapunov equations.
  - Solutions strategies via extending the ADI iteration to bilinear systems and EKSM as well as using preconditioned iterative solvers like CG or BiCGstab up to dimensions $n \sim 500,000$ in MATLAB®.
  - Optimal choice of shift parameters for ADI is a nontrivial task.
  - Existence of low-rank solutions in case of $A_i$ being full rank?
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  - Optimal choice of shift parameters for ADI is a nontrivial task.
  - Existence of low-rank solutions in case of $A_i$ being full rank?
- $H_2$ optimal model reduction:
  - Yields competitive approach, proven in industrial context.
  - Still high offline cost (= time for generating reduced-order model).
  - May need to switch to one-sided projection ($W = V$) to preserve stability.
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