# ITERATIVE SOLUTION OF LARGE-SCALE ALGEBRAIC RICCATI EQUATIONS WITH INDEFINITE HESSIAN

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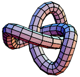

GAMM Jahrestagung
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# Goal

Derive numerical algorithms for solving

**(continuous-time) algebraic Riccati equation (ARE)**

with indefinite Hessian,

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0,$$

where

- $A \in \mathbb{R}^{n \times n}$ is large and sparse,
- $B_j \in \mathbb{R}^{n \times m_j}$ $(j = 1, 2)$,
- $C \in \mathbb{R}^{p \times n}$,
- $n \gg m_j, p$.

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## Hessian of $\mathcal{R}(X)$

Frechét derivative of $\mathcal{R}(\,.\,)$ at $X$:

$$\mathcal{R}'_X : Z \rightarrow (A + GX)^T Z + Z(A + GX).$$

Hessian/2nd order Frechét derivative of $\mathcal{R}(\,.\,)$ at $X$:

$$\mathcal{H} : (Z, Y) \rightarrow ZGY + YGZ$$

is indefinite in general unless $B_1 = 0$ or $B_2 = 0$.

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# Overview

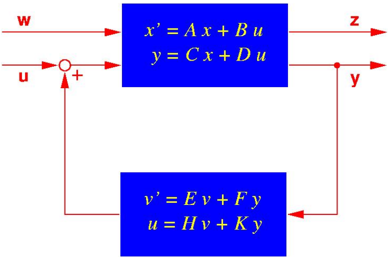
# Motivation: $H_\infty$-Control

## Linear time-invariant systems

$$\Sigma : \begin{cases} \dot{x} &= Ax + B_1 w + B_2 u, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{cases}$$

where $A \in \mathbb{R}^{n\times n}$, $B_k \in \mathbb{R}^{n\times m_k}$, $C_j \in \mathbb{C}^{p_j\times n}$, $D_{jk} \in \mathbb{R}^{p_j\times m_k}$.

$x$ – states of the system,
$w$ – exogenous inputs
$u$ – control inputs,
$z$ – performance outputs
$y$ – measured outputs



# Motivation: $H_\infty$-Control
## Transfer functions

Laplace transform $\Longrightarrow$ transfer function (in frequency domain)

$$G(s) = \left[\begin{array}{cc} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{array}\right] \equiv \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right].$$

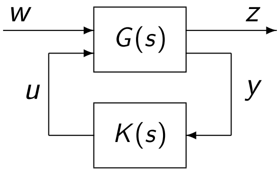
where for $x(0) = 0$, $G_{ij}$ are the rational matrix functions

- $G_{11}(s) = C_1(sI - A)^{-1}B_1 + D_{11}$,
- $G_{12}(s) = C_1(sI - A)^{-1}B_2 + D_{12}$,
- $G_{21}(s) = C_2(sI - A)^{-1}B_1 + D_{21}$,
- $G_{22}(s) = C_2(sI - A)^{-1}B_2 + D_{22}$,

describing the transfer from inputs to outputs of $\Sigma$ via

$$\begin{aligned} z(s) &= G_{11}(s)w(s) + G_{12}(s)u(s), \\ y(s) &= G_{21}(s)w(s) + G_{22}(s)u(s). \end{aligned}$$

Consider closed-loop system, where $K(s)$ is an internally stabilizing controller, i.e., $K$ stabilizes $G$ for $w \equiv 0$.



# Motivation: $H_\infty$-Control

## The $H_\infty$-Optimization Problem

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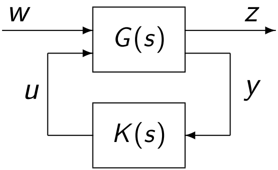


**Goal:**

find $K$ that minimize error outputs

$$z = \left(G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}\right) w =: \mathcal{F}(G, K)w,$$

where $\mathcal{F}(G, K)$ is the linear fractional transformation of $G$, $K$.

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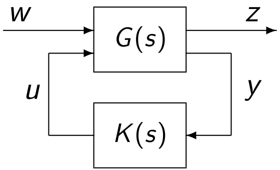


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**$H_\infty$-optimal control problem:**

$$\min_{K \text{ stabilizing}} \|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty}.$$

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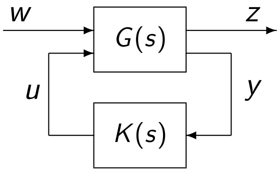


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**$H_\infty$-suboptimal control problem:**

For given constant $\gamma > 0$, find all internally stabilizing controllers satisfying

$$\|\mathcal{F}(G, K)\|_{\mathcal{H}_\infty} < \gamma.$$

## Simplifying assumptions

1. $D_{11} = 0$;
2. $D_{22} = 0$;
3. $(A, B_1)$ stabilizable, $(C_1, A)$ detectable;
4. $(A, B_2)$ stabilizable, $(C_2, A)$ detectable ($\Longrightarrow \Sigma$ internally stabilizable);
5. $D_{12}^T [\, C_1 \;\; D_{12} \,] = [\, 0 \;\; I_{m_2} \,]$;
6. $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix}$.

Remark. 1.,2.,5.,6. only for notational convenience, 3. can be relaxed, but derivations get even more complicated.

# Motivation: $H_\infty$-Control

## Solution of the $H_\infty$-(Sub-)Optimal Control Problem

**Theorem** [Doyle/Glover/Khargonekar/Francis '89]

Given the Assumptions 1.–6., there exists an admissible controller $K(s)$ solving the $H_\infty$-suboptimal control problem $\Longleftrightarrow$

(i) There exists a solution $X_\infty = X_\infty^T \geq 0$ to the ARE

$$C_1 C_1^T + A^T X + XA + X(\gamma^{-2} B_1 B_1^T - B_2 B_2^T)X = 0, \qquad (1)$$

such that $A_X$ is Hurwitz, where $A_X := A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty$.

(ii) There exists a solution $Y_\infty = Y_\infty^T \geq 0$ to the ARE

$$B_1 B_1^T + AY + YA^T + Y(\gamma^{-2} C_1 C_1^T - C_2 C_2^T)Y = 0, \qquad (2)$$

such that $A_Y$ is Hurwitz where $A_Y := A + Y_\infty(\gamma^{-2} C_1 C_1^T - C_2 C_2^T)$.

(iii) $\gamma^2 > \rho(X_\infty Y_\infty)$.

**$H_\infty$-optimal control**

Find minimal $\gamma$ for which (i)–(iii) are satisfied $\rightsquigarrow$ $\gamma$-iteration based on solving AREs (1)–(2) repeatedly for different $\gamma$.

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## $H_\infty$-(sub-)optimal controller

If (i)–(iii) hold, a suboptimal controller is given by

$$\hat{K}(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & 0 \end{array}\right] = \hat{C}(sI_n - \hat{A})^{-1}\hat{B},$$

where for

$$Z_\infty := (I - \gamma^{-2} Y_\infty X_\infty)^{-1},$$

$$\begin{aligned} \hat{A} &:= A + (\gamma^{-2} B_1 B_1^T - B_2 B_2^T) X_\infty - Z_\infty Y_\infty C_2^T C_2, \\ \hat{B} &:= Z_\infty Y_\infty C_2^T, \\ \hat{C} &:= -B_2^T X_\infty. \end{aligned}$$

$\hat{K}(s)$ is the central or minimum entropy controller.

# Solving Large-Scale Standard AREs

General form for $A, G = G^T, W = W^T \in \mathbb{R}^{n\times n}$ given and $X \in \mathbb{R}^{n\times n}$ unknown:

$$0 \quad = \quad \mathcal{R}(X) := A^T X + XA - XGX + W.$$

Large-scale AREs from semi-discretized PDE control problems:

- $n = 10^3 - 10^6$ ($\Longrightarrow 10^6 - 10^{12}$ unknowns!),
- $A$ has sparse representation ($A = -M^{-1}K$ for FEM),
- $G, W$ low-rank with $G, W \in \{BB^T, C^TC\}$, where $B \in \mathbb{R}^{n\times m}$, $m \ll n$, $\quad C \in \mathbb{R}^{p\times n}$, $p \ll n$.
- Standard (eigenproblem-based) $\mathcal{O}(n^3)$ methods are not applicable!

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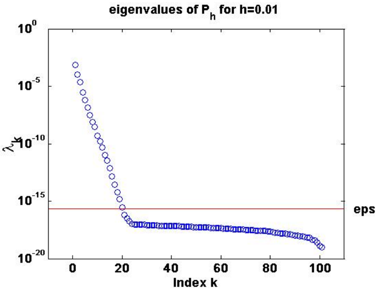
# Low-Rank Approximation

ARE $0 = A^T X + XA - XBB^T X + CC^T$

Consider spectrum of ARE solution (analogous for Lyapunov equations).

Example:

- Linear 1D heat equation with point control,
- $\Omega = [0, 1]$,
- FEM discretization using linear B-splines,
- $h = 1/100 \implies n = 101$.




Idea: $X = X^T \geq 0 \implies$

$$X = YY^T = \sum_{k=1}^{n} \lambda_k y_k y_k^T \approx Y^{(r)}(Y^{(r)})^T = \sum_{k=1}^{r} \lambda_k y_k y_k^T.$$

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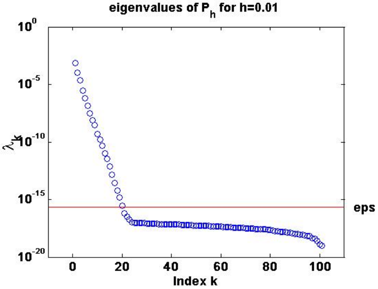


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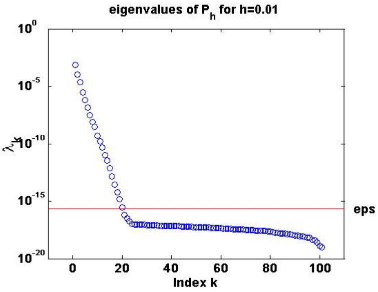


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# Newton's Method for AREs

[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

- Consider $0 = \mathcal{R}(X) = C^T C + A^T X + XA - XBB^T X.$
- Frechét derivative of $\mathcal{R}(X)$ at $X$:

$$\mathcal{R}'_X : Z \to (A - BB^T X)^T Z + Z(A - BB^T X).$$

- Newton-Kantorovich method:

$$X_{j+1} = X_j - \left(\mathcal{R}'_{X_j}\right)^{-1} \mathcal{R}(X_j), \quad j = 0, 1, 2, \ldots$$

**Newton's method (with line search) for AREs**

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T X_j =: A - BK_j$.
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$.
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- Consider $\quad 0 = \mathcal{R}(X) = C^TC + A^TX + XA - XBB^TX.$
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# Newton's Method for AREs

## Properties and Implementation



- Convergence for $K_0$ stabilizing:
  - $A_j = A - BK_j = A - BB^T X_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(X_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} X_j = X_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver; here, ADI iteration: linear systems with dense, but "sparse+low rank" coefficient matrix $A_j$:

  $A_j \quad = \quad A \quad - \quad B \quad \cdot \quad K_j$

  $\quad = \quad$ [sparse] $\quad - \quad$ [$m$] $\quad \cdot \quad$ [ ]

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $X = X^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

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  $A_j \;=\; A \;-\; B \;\cdot\; K_j$

  $\;\;\;\;\;=\;$ sparse $\;-\;$ $m$ $\;\cdot\;$

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- BUT: $X = X^T \in \mathbb{R}^{n\times n} \implies n(n+1)/2$ unknowns!

# ADI Method for Lyapunov Equations



- For $A \in \mathbb{R}^{n \times n}$ stable, $B \in \mathbb{R}^{n \times m}$ ($w \ll n$), consider Lyapunov equation

$$AX + XA^T = -BB^T.$$

- ADI Iteration: [Wachspress 1988]

$$\begin{aligned}(A + p_k I)X_{(k-1)/2} &= -BB^T - X_{k-1}(A^T - p_k I)\\ (A + \overline{p_k} I){X_k}^T &= -BB^T - X_{(k-1)/2}(A^T - \overline{p_k} I)\end{aligned}$$

with parameters $p_k \in \mathbb{C}^-$ and $p_{k+1} = \overline{p_k}$ if $p_k \notin \mathbb{R}$.

- For $X_0 = 0$ and proper choice of $p_k$: $\lim_{k \to \infty} X_k = X$ superlinear.
- Re-formulation using $X_k = Y_k Y_k^T$ yields iteration for $Y_k$...

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# Factored ADI Iteration

Lyapunov equation $0 = AX + XA^T = -BB^T$.



Setting $X_k = Y_k Y_k^T$, some algebraic manipulations $\Longrightarrow$

**Algorithm** [Penzl 1997, Li/White 2002, B./Li/Penzl 1999/2006]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Y_1 \leftarrow V_1$$

FOR $j = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1}V_{k-1}\right),$$

$$Y_k \leftarrow \left[\ Y_{k-1} \quad V_k\ \right]$$

$$Y_k \leftarrow \mathrm{rrqr}(Y_k, \tau) \qquad \text{\% column compression}$$

At convergence, $Y_{k_{\max}} Y_{k_{\max}}^T \approx X$, where

$$\mathrm{range}\,(Y_{k_{\max}}) = \mathrm{range}\,(\left[\ V_1 \quad \ldots \quad V_{k_{\max}}\ \right]), \qquad V_k = \left[\ \right] \in \mathbb{C}^{n \times m}.$$

**Note:** Implementation in real arithmetic possible by combining two steps.

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# Low-Rank Newton-ADI for AREs



Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(X_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + \underbrace{(X_j + N_j)}_{=X_{j+1}} A_j = \underbrace{-C^T C - X_j B B^T X_j}_{=:-W_j W_j^T}$$

Set $X_j = Z_j Z_j^T$ for $\operatorname{rank}(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

**Factored Newton Iteration** [B./Li/Penzl 1999/2008]

Solve Lyapunov equations for $Z_{j+1}$ directly by factored ADI iteration and use 'sparse + low-rank' structure of $A_j$.

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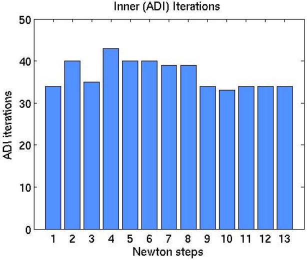
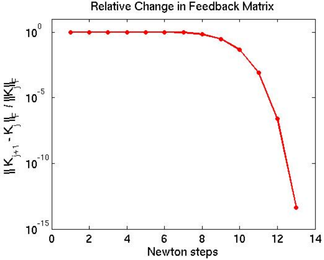
Solve Lyapunov equations for $Z_{j+1}$ directly by factored ADI iteration and use 'sparse + low-rank' structure of $A_j$.

# Solving Large-Scale Standard AREs
## Numerical Results



- Linear 2D heat equation with homogeneous Dirichlet boundary and point control/observation.
- FD discretization on uniform $150 \times 150$ grid.
- $n = 22.500$, $m = p = 1$, 10 shifts for ADI iterations.
- Convergence of large-scale matrix equation solvers:



## Performance of Newton's method for accuracy $\sim 1/n$

| grid | unknowns | $\frac{\Vert\mathcal{R}(X)\Vert_F}{\Vert X\Vert_F}$ | it. (ADI it.) | CPU (sec.) |
|---|---|---|---|---|
| $8 \times 8$ | 2,080 | 4.7e-7 | 2 (8) | 0.47 |
| $16 \times 16$ | 32,896 | 1.6e-6 | 2 (10) | 0.49 |
| $32 \times 32$ | 524,800 | 1.8e-5 | 2 (11) | 0.91 |
| $64 \times 64$ | 8,390,656 | 1.8e-5 | 3 (14) | 7.98 |
| $128 \times 128$ | 134,225,920 | 3.7e-6 | 3 (19) | 79.46 |

Here,

- Convection-diffusion equation,
- $m = 1$ input and $p = 2$ outputs,
- $X = X^T \in \mathbb{R}^{n\times n} \Rightarrow \frac{n(n+1)}{2}$ unknowns.

# AREs with Indefinite Hessian

Back to

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0.$$

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- For large-scale problems, resulting, e.g., from $H_\infty$ control, standard methods based on Hamiltonian/even eigenvalue problem can not be used due to $\mathcal{O}(n^3)$ complexity/dense matrix algebra.
- Krylov subspace methods might be employed, but so far no convergence results, and in case of convergence, no guarantee that stabilizing solution is computed.
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# AREs with Indefinite Hessian



Back to

$$\mathcal{R}(X) := C^T C + A^T X + XA + X(B_1 B_1^T - B_2 B_2^T)X = 0.$$

## Problems

Quick-and-dirty solution: consider $X^{-1}\mathcal{R}(X)X^{-1} = 0$ [Damm 2002]

$\rightsquigarrow$ standard ARE for $\tilde{X} \equiv X^{-1}$

$$\tilde{\mathcal{R}}(\tilde{X}) := (B_1 B_1^T - B_2 B_2^T) + \tilde{X}A^T + A\tilde{X} + \tilde{X}C^T C\tilde{X} = 0.$$

Newton's method will converge to stabilizing solution, Newton-ADI can be employed (with modification for indefinite constant term).

But: low-rank approximation of $\tilde{X}$ will not yield good approximation of $X$ $\Rightarrow$ not feasible for large-scale problems!

# Lyapunov Iterations/Perturbed Hessian Approach

[Cherfi/Abou-Kandil/Bourles 2005 (Proc. ACSE 2005)]

## Idea

Perturb Hessian to enforce semi-definiteness: write

$$0 = A^T X + XA + Q - XGX = A^T X + XA + Q - XDX + X(D-G)X,$$

where $D = G + \alpha I \geq 0$ with $\alpha \geq \min\{0, -\lambda_{\max}(G)\}$.

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Here: $G = B_2 B_2^T - B_1 B_1^T$

$\Rightarrow$ use $\alpha = \|B_1\|^2$ for spectral/Frobenius norm or

$$\alpha = \|B_1\|_1 \cdot \|B_1\|_\infty.$$

## Remark

$W \geq -G$ can be used instead of $\alpha I$, e.g., $W = \beta B_1 B_1^T$ with $\beta \geq 1$.

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## Lyapunov iteration

Based on

$$(A - DX)^T X + X(A - DX) = -Q - XDX - \alpha X^2,$$

iterate

FOR $k = 0, 1, \ldots$, solve Lyapunov equation

$$(A - DX_k)^T X_{k+1} + X_{k+1}(A - DX_k) = -Q - X_k DX_k - \alpha X_k^2.$$

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Easy to convert to low-rank iteration employing low-rank ADI for Lyapunov equations, e.g. with $W = B_1 B_1^T$ instead of $\alpha I$: the Lyapunov equation becomes

$$\begin{aligned}(A - B_2 B_2^T Y_k Y_k)^T Y_{k+1} Y_{k+1}^T &+ Y_{k+1} Y_{k+1}^T (A - B_2 B_2^T Y_k Y_k) \\ &= -CC^T - Y_k Y_k^T B_1 B_1^T Y_k Y_k^T - Y_k Y_k^T B_2 B_2^T Y_k Y_k^T \\ &= -[\, C,\ Y_k Y_k^T B_1,\ Y_k Y_k^T B_2 \,] \begin{bmatrix} C^T \\ B_1^T Y_k Y_k^T \\ B_2^T Y_k Y_k^T \end{bmatrix}.\end{aligned}$$

# Lyapunov Iterations/Perturbed Hessian Approach

## Convergence

**Theorem** [Cherfi/Abou-Kandil/Bourles 2005]

If

- $\exists\ \hat{X}$ such that $\mathcal{R}(\hat{X}) \geq 0$,
- $\exists\ X_0 = X_0^T \geq \hat{X}$ such that $\mathcal{R}(X_0) \leq 0$ and $A - DX_0$ is Hurwitz,

then

a) $X_0 \geq \ldots \geq X_k \geq X_{k+1} \geq \ldots \geq \hat{X}$,

b) $\mathcal{R}(X_k) \leq 0$ for all $k = 0, 1, \ldots$,

c) $A - DX_k$ is Hurwitz for all $k = 0, 1, \ldots$,

d) $\exists \quad \lim_{k\to\infty} X_k =: \underline{X} \geq \hat{X}$,

e) $\underline{X}$ is semi-stabilizing.

**Main problems**

- Conditions for initial guess make its computation difficult.
- Observed convergence is linear.

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# Riccati Iterations
[LANZON/FENG/B.D.O. ANDERSON 2007 (Proc. ECC 2007)]

## Idea

Consider

$$A^T X + XA + C^T C + X(B_1 B_1^T - B_2 B_2^T)X =: \mathcal{R}(X).$$

Then

$$\begin{aligned}\mathcal{R}(X+Z) &= \mathcal{R}(X) + (\underbrace{A + (B_1 B_1^T - B_2 B_2^T)X}_{=:\widehat{A}})^T Z + Z\widehat{A} \\ &\quad + Z(B_1 B_1^T - B_2 B_2^T)Z.\end{aligned}$$

Furthermore, if $X = X^T$, $Z = Z^T$ solve the standard ARE

$$0 = \mathcal{R}(X) + \widehat{A}^T Z + Z\widehat{A} - ZB_2 B_2^T Z,$$

then

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# Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

## Riccati iteration

1. Set $X_0 = 0$.
2. FOR $k = 1, 2, \ldots,$
   (i) Set $A_k := A + B_1(B_1^T X_k) - B_2(B_2^T X_k)$.
   (ii) Solve the ARE
   $$\mathcal{R}(X_k) + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0.$$
   (iii) Set $X_{k+1} := X_k + Z_k$.
   (iv) IF $\|B_1^T Z_k\|_2 < \text{tol}$ THEN Stop.

Remark. ARE for $k = 0$ is the standard LQR/$H_2$ ARE.

# Riccati Iterations

[Lanzon/Feng/B.D.O. Anderson 2007 (Proc. ECC 2007)]

**Theorem** [Lanzon/Feng/B.D.O. Anderson 2007]

If

- $(A, B_2)$ stabilizable,
- $(A, C)$ has no unobservable purely imaginary modes, and
- $\exists$ stabilizing solution $X_-$,

then

a) $(A + B_1 B_1^T X_k, B_2)$ stabilizable for all $k = 0, 1, \ldots$,

b) $Z_k \geq 0$ for all $k = 0, 1, \ldots$,

c) $A + B_1 B_1^T X_k - B_2 B_2^T X_{k+1}$ is Hurwitz for all $k = 0, 1, \ldots$,

d) $\mathcal{R}(X_{k+1}) = Z_k B_1 B_1^T Z_k$ for all $k = 0, 1, \ldots$,

e) $X_- \geq \ldots \geq X_{k+1} \geq X_k \geq \ldots \geq 0$.

f) If $\exists \lim_{k\to\infty} X_k =: \underline{X}$, then $\underline{X} = X_-$, and

g) convergence is locally quadratic.

## Riccati iteration – low-rank version [B. 2008]

1 Solve the ARE

$$C^T C + A^T Z_0 + Z_0 A - Z_0 B_2 B_2^T Z_0 = 0$$

using Newton-ADI, yielding $Y_0$ with $Z_0 \approx Y_0 Y_0^T$.

2 Set $R_1 := Y_0$. $\{\% \ R_1 R_1^T \approx X_1.\}$

3 FOR $k = 1, 2, \ldots,$

(i) Set $A_k := A + B_1(B_1^T R_k)R_k^T - B_2(B_2^T R_k)R_k^T$.

(ii) Solve the ARE

$$Y_{k-1}(Y_{k-1}^T B_1)(B_1^T Y_{k-1})Y_{k-1}^T + A_k^T Z_k + Z_k A_k - Z_k B_2 B_2^T Z_k = 0$$

using Newton-ADI, yielding $Y_k$ with $Z_k \approx Y_k Y_k^T$.

(iii) Set $R_{k+1} := \texttt{rrqr}([\, R_k, \ Y_k \,], \tau)$. $\{\% \ R_{k+1} R_{k+1}^T \approx X_{k+1}\}$

(iv) IF $\|(B_1^T Y_k) Y_k^T\|_2 <$ tol THEN Stop.

# AREs with Indefinite Hessian
## Numerical example



- Trivial example ($n = 2$) from [Cherfi/Abou-Kandil/Bourles 2005].
- Compare convergence of Lyapunov and Riccati iterations.
- Solution of standard AREs with Newton's method.

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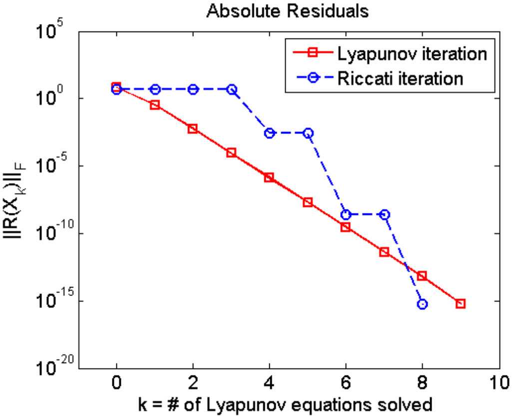

# Conclusions and Open Problems



- Low-rank Riccati iteration yields a reliable and efficient method for large-scale AREs with indefinite Hessian, useful, e.g., for $H_\infty$ optimization of PDE control problems.
- Low-rank Lyapunov iteration is an extremely simple variant for large-scale problems, but exhibits slower convergence and requires difficult-to-compute initial value.
- To-Do list:
  - Implement Riccati iteration in LyaPack/MESS style.
  - More numerical tests.
  - Re-write Riccati iteration as feedback iteration.
  - Efficient computation of initial value for Lyapunov iterations?
  - $\exists$ perturbed Hessian so that Lyapunov iteration quadratically convergent?

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**Fin.**