BALANCING-RELATED MODEL REDUCTION FOR LARGE-SCALE UNSTABLE SYSTEMS

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Overview

1 Motivation

2 Model Reduction Based on Balancing
- Balancing-Related Model Reduction

3 Solving Large-Scale Algebraic Riccati Equations
- Newton's Method
- Low-Rank Newton-Kleinman for AREs
- Algebraic Bernoulli Equations

4 Numerical Results

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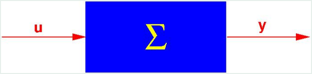
The Model Reduction Problem

Original System

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t). \end{cases}$$

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.



Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{A}\hat{x}(t) + \widehat{B}u(t), \\ \hat{y}(t) = \widehat{C}\hat{x}(t) + \widehat{D}u(t). \end{cases}$$

- states $\hat{x}(t) \in \mathbb{R}^r$, $r \ll n$
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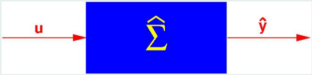

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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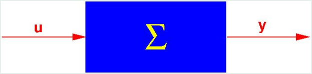

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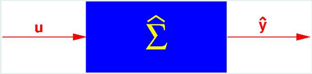

Goal:

$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\|$ for all admissible input signals.

Balancing-Related Model Reduction

Basic Principle of Balanced Truncation (BT)

Given positive semidefinite selfadjoint operators $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n, \ldots) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq \ldots \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical BT

Mullis/Roberts '76, Moore '81

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) — Jonckheere/Silverman '83

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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H_∞ balanced truncation (HinfBT) — Mustafa/Glover '91

- Based on H_∞ theory.
- P, Q solve dual H_∞-AREs

$$\begin{aligned} 0 &= AP + PA^T - (1-\delta)PC^T CP + B^T B, \\ 0 &= A^T Q + QA - (1-\delta)QBB^T Q + C^T C, \end{aligned}$$

where $\delta = 1 - \frac{1}{\gamma^2}$ and $\gamma > \gamma_{\text{opt}} := \min_{\mathcal{K}\text{ stabilizing}} \|\mathcal{F}(G, \mathcal{K})\|_{H_\infty}$.

Properties of Balancing-Related Model Reduction

applied to minimal systems

LQG Balanced Truncation

- Applicable to unstable systems.
- Directly "applicable" to ∞-dim. systems, developed convergence theory [Curtain '03].
- Provides LQG controller as by-product which exponentially stabilizes ∞-dim. plant [Morris '94].
- Computable error bound:

$$\| \begin{bmatrix} N & M \end{bmatrix} - \begin{bmatrix} \hat{N} & \hat{M} \end{bmatrix} \|_\infty \leq \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1 + (\sigma_j^{LQG})^2}},$$

where $G = M^{-1}N$, $G = \hat{M}^{-1}\hat{N}$ are left coprime factorizations of G, $\hat{G}$.

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where $G = M^{-1}N$, $G = \hat{M}^{-1}\hat{N}$ are left coprime factorizations of G, $\hat{G}$ and

$$\beta := \sqrt{1 - \frac{1}{\gamma^2}}.$$

(Need to assume $\gamma > \max\{1, \gamma_{\text{opt}}\}$.)

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H_∞ Balanced Truncation

Provides H_∞ controller with *a priori* robustness check as by-product: if $\widehat{\mathcal{K}}$ is the H_∞-(minimum entropy) controller of the (normalized) reduced-order system, then $\widehat{\mathcal{K}}$ stabilizes $G \iff$

$$\varepsilon^{H_\infty} := \delta^{H_\infty}(\gamma + \beta) < 1.$$

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In this case,

$$\mathcal{F}(G, \widehat{\mathcal{K}}) < \gamma + \frac{\delta^{H_\infty}(1+\gamma)(1+\gamma+\beta)}{1-\varepsilon^{H_\infty}}.$$

Computation of Reduced-Order Systems from Gramians

1. Given the Gramians P, Q of the n-dimensional system from either pair of LEs or AREs in factorized form

$$P = S^T S, \quad Q = R^T R,$$

compute SVD

$$SR^T = [\, U_1,\ U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$$

2. Set $W = R^T V_1 \Sigma_1^{-1/2}$ and $V = S^T U_1 \Sigma_1^{-1/2}$.
3. Then the reduced-order model is

$$(A_r, B_r, C_r) = (W^T A V, W^T B, C V).$$

Thus, need to solve large-scale algebraic Riccati equations—but need only factors!

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MOR for Unstable Systems

Peter Benner

Motivation

Model Reduction Based on Balancing

Large AREs

Newton's Method

Low-Rank Newton-Kleinman for AREs

Algebraic Bernoulli Equations

Numerical Results

Conclusions and Open Problems

- Consider $0 = \mathcal{R}(Q) = C^T C + A^T Q + QA - QBB^T Q.$
- Frechét derivative of $\mathcal{R}(Q)$ at Q:

$$\mathcal{R}_Q' : Z \to (A - BB^T Q)^T Z + Z(A - BB^T Q).$$

- Newton-Kantorovich method:

$$Q_{j+1} = Q_j - \left(\mathcal{R}_{Q_j}'\right)^{-1} \mathcal{R}(Q_j), \quad j = 0, 1, 2, \ldots$$

Newton's method (with line search) for AREs

FOR $j = 0, 1, \ldots$

1. $A_j \leftarrow A - BB^T Q_j =: A - BK_j.$
2. Solve the Lyapunov equation $A_j^T N_j + N_j A_j = -\mathcal{R}(Q_j).$
3. $Q_{j+1} \leftarrow Q_j + t_j N_j.$

END FOR j

Solving Large-Scale Algebraic Riccati Equations

Newton's Method

[Kleinman '68, Mehrmann '91, Lancaster/Rodman '95, B./Byers '94/'98, B. '97, Guo/Laub '99]

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Newton's Method

Properties and Implementation

- Convergence for K_0 stabilizing:
  - $A_j = A - BK_j = A - BB^T Q_j$ is stable $\forall\ j \geq 0$.
  - $\lim_{j\to\infty} \|\mathcal{R}(Q_j)\|_F = 0$ (monotonically).
  - $\lim_{j\to\infty} Q_j = Q_* \geq 0$ (locally quadratic).
- Need large-scale Lyapunov solver for dense, but "sparse+low rank" coefficient matrix A_j:

$$A_j = A - B \cdot K_j$$

$$= \boxed{\text{sparse}} - \boxed{m} \cdot \boxed{}$$

- $m \ll n \implies$ efficient "inversion" using Sherman-Morrison-Woodbury formula:

$$(A - BK_j)^{-1} = (I_n + A^{-1}B(I_m - K_jA^{-1}B)^{-1}K_j)A^{-1}.$$

- Solve Lyapunov equations using low-rank ADI [Penzl '99, Li/White '02], K-PIK [Simoncini '06] or alike, yielding $Q_j = Z_jZ_j^T$, $Z_j \in \mathbb{R}^{n\times n_j}$, $n_j \ll n$.

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Low-Rank Newton-Kleinman for AREs

[Banks/Ito '91, B./Li/Penzl 1999]

Re-write Newton's method for AREs

$$A_j^T N_j + N_j A_j = -\mathcal{R}(Q_j)$$

$$\Longleftrightarrow$$

$$A_j^T \underbrace{(Q_j + N_j)}_{=Q_{j+1}} + \underbrace{(Q_j + N_j)}_{=Q_{j+1}} A_j = \underbrace{-C^T C - Q_j B B^T Q_j}_{=:-W_j W_j^T}$$

Set $Q_j = Z_j Z_j^T$ for $\mathrm{rank}\,(Z_j) \ll n \Longrightarrow$

$$A_j^T \left(Z_{j+1} Z_{j+1}^T\right) + \left(Z_{j+1} Z_{j+1}^T\right) A_j = -W_j W_j^T$$

Low-Rank Newton-Kleinman for AREs

Computing stabilizing initial feedback

Consider ARE:

$$A_j^T \underbrace{(X_j + N_j)}_{=X_{j+1}} + X_{j+1} \underbrace{(A - BB^T X_j)}_{=:A_j} = -M_h - X_j BB^T X_j \quad \text{for } j = 1, 2, \ldots$$

Recall: for convergence to stabilizing solution need

$A_0 := A - BB^T X_0$ stable, i.e., all eigenvalues in left half plane.

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Basically, 3 approaches to compute $K_0 := B^T X_0$:

- pole placement,
- Bass algorithm (based on Lyapunov equation) [Armstrong '75],
- algebraic Bernoulli equations: [B. '06/'07], (for discrete-time systems: [Gallivan/Rao/Van Dooren 06]).

Algebraic Bernoulli Equations

Consider the algebraic Bernoulli equation

$$A^T X + XA - XBB^T X = 0$$

associated to a standard ARE with zero constant term.

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Theorem

a) Let (A, B) be controllable. Then

- there exist symmetric solutions $X_+ \geq 0$, $X_- \leq 0$, with $X_- \leq X \leq X_+$ for all solutions X of the ABE;
- X_- is the unique solution satisfying $\Lambda(A - BB^T X_-) \subset \mathbb{C}^+ \cup \imath\mathbb{R}$;
- X_+ is the unique solution satisfying $\Lambda(A - BB^T X_+) \subset \mathbb{C}^- \cup \imath\mathbb{R}$.
- If $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then X_- is the unique anti-stabilizing solution and X_+ is the unique stabilizing solution of the ABE.

b) If (A, B) is stabilizable and $\Lambda(A) \cap \imath\mathbb{R} = \emptyset$, then the ABE has a unique stabilizing solution X_+ and $X_+ \geq 0$.

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$$\operatorname{rank}(X_+) = k,$$

where k is the number of eigenvalues of A in $\mathbb{C}^+$.

Hence,

$$X_+ = Y_+ Y_+^T, \quad \text{where} \quad Y_+ \in \mathbb{R}^{n \times k}.$$

Theorem [B. '07]

$$\Lambda(A - BB^T X_+) = (\Lambda(A) \cap \mathbb{C}^-) \cup -(\Lambda(A) \cap \mathbb{C}^+),$$

i.e., unstable eigenvalues are reflected at imaginary axis.

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Computation of X_+

- Solve as ARE (inefficient).
- Sign function method [Barrachina/B./Quintana-Ortí '05].
- Sign function method for Y_+ [B. '06, Barr./B./Q.-Ortí '07].
- Extension to descriptor systems [B. '08].
- For large-scale systems, use partial stabilization idea:
  1. Project onto unstable invariant/deflating subspace of $A/\lambda E - A$,

     $$\tilde{Q}^T A \tilde{Q} = \tilde{A} \in \mathbb{R}^{k \times k}, \quad \text{set } \tilde{B} := \tilde{Q}^T B.$$

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Cf. also related work by [Amodei/Bouchon '08].

Algebraic Bernoulli Equations

Consider the algebraic Bernoulli equation

$$A^T X + XA - XBB^T X = 0$$

associated to a standard ARE with zero constant term.

Computation of X_+

- Solve as ARE (inefficient).
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Numerical Results

- Point control problem for 1D heat flow.
- FEM $\rightsquigarrow$ $n = 200$, $m = 1$, $p = 1$.
- $\gamma_{\mathrm{opt}} \approx 5.938$, $\gamma = 6$.
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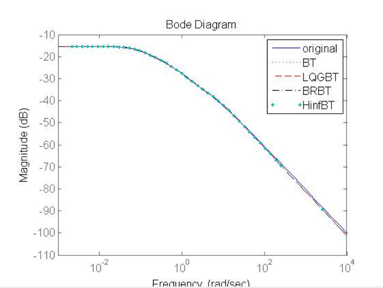
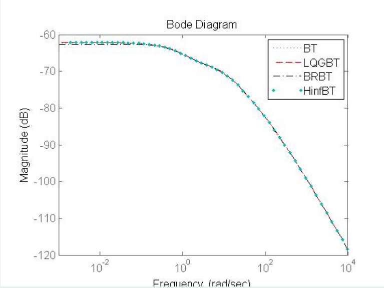
Bode plot



Absolute errors



Numerical Results

Algebraic Bernoulli Equations: BT vs. LQG BT

- Boundary control problem for 2D heat flow in copper on rectangular domain; control acts on two sides via Robins BC.
- FDM $\rightsquigarrow$ $n = 4496,\ m = 2;$ 4 sensor locations $\rightsquigarrow$ $p = 4.$
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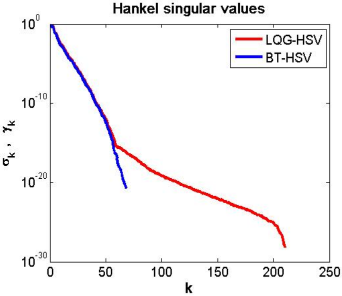

Source: *COMPl$_e$ib* v1.1, `www.compleib.de`.

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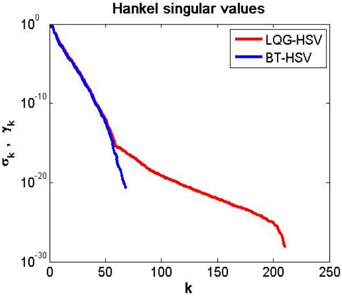

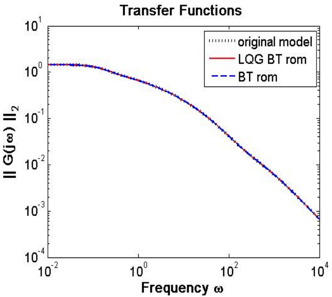

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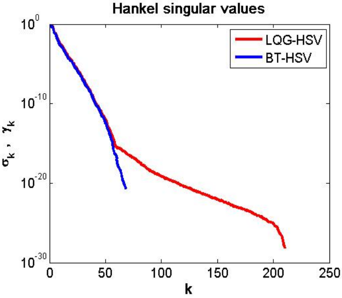

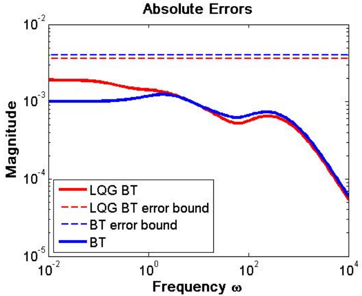

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Conclusions and Open Problems

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