Doubly Structured Polar Decompositions and Algebraic Riccati Equations

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Overview

Polar decomposition in $\mathbb{C}^n$

Let $A \in \mathbb{C}^{n \times n}$, then

$$A = UM, \quad U^{-1} = U^* \text{ (unitary)}, \; M = M^* \geq 0,$$

is called a polar decomposition of A.

Note: any matrix admits a polar decomposition as

$$A = (UV^*)(V \Sigma V^*),$$

where $A = U \Sigma V^*$ is the SVD of A.

Given $A \in \mathbb{C}^{n\times n}$, $H = H^* \in \mathbb{C}^{n\times n}$ nonsingular, and the corresponding (indefinite) inner product

$$\langle x, y\rangle_H := \langle Hx, y\rangle$$

where $\langle\, .\, ,\, .\, \rangle$ is the standard unitary inner product, then the *H*-adjoint of M, i.e., the adjoint of M w.r.t. $\langle\, .\, ,\, .\, \rangle_H$, is $M^H = H^{-1}M^*H$.

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H-polar decomposition

$$A = UM, \quad U^{-1} = U^H \text{ (H-unitary)}, \; M = M^H.$$

Doubly Structured Polar Decompositions

Generalization of polar decompositions in finite-dim. indefinite inner product spaces

Given $A \in \mathbb{C}^{n\times n}$, $H = H^* \in \mathbb{C}^{n\times n}$ nonsingular, and the corresponding (indefinite) inner product

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H-polar decomposition

$$A = UM, \quad U^{-1} = U^H \text{ (H-unitary)}, \;\; M = M^H.$$

Note: not all $A \in \mathbb{C}^{n\times n}$ admit an H-polar decomposition!
Existence results:

- Bolshakov, van der Mee, Ran, Reichstein, Rodman (1997)
- Lins, Meade, Mehl, Rodman (2001)
- Kintzel (2003,2005)
- Mehl, Ran, Rodman (2006)

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(G, H)-polar decomposition [Kintzel 2003/2005]

Let $H = H^*, G = G^*$ nonsingular, then

$$A = UM, \quad U^{-1} = U^G = U^H, \; M = M^G = M^H,$$

is a (G, H)-polar decomposition. In this case

- U is (G, H)-unitary,
- M is (G, H)-selfadjoint.

(G, H)-polar decomposition is H-semidefinite if $HM \geq 0$.

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Unitary (Orthogonal) Procrustes Problems

Given $C, B \in \mathbb{C}^{m\times n}$, find $U \in \mathbb{C}^{m\times m}$ unitary minimizing

$$\|UC - B\|_F.$$

In other words, for $C = [c_1, \dots, c_n], B = [b_1, \dots, b_n]$, minimize

$$\sum_{k=1}^{n} \langle Uc_k - b_k, Uc_k - b_k \rangle.$$

under the constraint $U^{-1} = U^*$.

Solution: $U =$ unitary factor of polar decomposition $BC^* = UM$.

(G, H)-Isometric Procrustes Problem [KINTZEL 2003/2005]

Given $C = [c_1, \ldots, c_n], B = [b_1, \ldots, b_n] \in \mathbb{C}^{m \times n}$, find $U \in \mathbb{C}^{m \times m}$ optimizing

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Solution (for $H^{-1}G = \mu^2 G^{-1}H$, $\mu \in \mathbb{R} \setminus \{0\}$):

existence $\iff$ there exists an H-semidefinite (G, H)-polar decomposition

$$A := BC^*H + G^{-1}HBC^*G = UM.$$

Then, the optimizing U is just the (G, H)-unitary factor.

Recall: want $A = UM$ so that $U^{-1} = U^G = U^H$ and $M = M^G = M^H$.

Necessary condition

$A^H = A^G$, as $A^H = M^H U^H = M^G U^G = A^G$.

Note: if $\lambda H - G \in \mathbb{C}^{n \times n}$ is non-defective Hermitian matrix pencil, such matrices impose a "normal" form of (A, H, G):

$$(S^{-1}AS, S^* HS, S^* GS) = (A_1 \oplus \ldots \oplus A_k, H_1 \oplus \ldots \oplus H_k, G_1 \oplus \ldots \oplus G_k),$$

where

for real eigenvalues μ_j, $j = 1, \ldots, r$, of $\lambda H - G$:

$$A_j \in \mathbb{C}^{p_j \times p_j}, \quad H_j = I_{p_j - q_j} \oplus -I_{q_j}, \quad G_j = \mu_j (I_{p_j - q_j} \oplus -I_{q_j}),$$

for non-real eigenvalues μ_j, $j = r+1, \ldots, \ell$, of $\lambda H - G$:

$$A_j = \begin{bmatrix} A_{j,1} & \\ & A_{j,2} \end{bmatrix} \in \mathbb{C}^{2p_j \times 2p_j}, \quad H_j = \begin{bmatrix} & I_{p_j} \\ I_{p_j} & \end{bmatrix}, \quad G_j = \begin{bmatrix} & \overline{\mu_j} I_{p_j} \\ \mu I_{p_j} & \end{bmatrix}.$$

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Necessary and sufficient condition

If $\lambda H - G \in \mathbb{C}^{n \times n}$ is non-defective Hermitian matrix pencil, then $A^H = A^G$ admits a (G, H) polar decomposition

$$\Longleftrightarrow$$

in the "normal" form of A,

1. all blocks A_j corresponding to real eigenvalues of $\lambda H - G$ admit an H_j-polar decomposition,
2. all blocks A_j corresponding to non-real eigenvalues of $\lambda H - G$ satisfy nonsymmetric algebraic Riccati equations (nARE)

$$A_{j,1} = U_j A_{j,2}^* U_j,$$

with $U_j \in \mathbb{C}^{p_j \times p_j}$ nonsingular.

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Nonsymmetric Algebraic Riccati Equations

General form of nARE:

$$0 = A + DX - XC - XBX,$$

where $A, B^* \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times m}$, $D \in \mathbb{C}^{n\times n}$ are given and $X \in \mathbb{C}^{n\times m}$ is unknown.

Corresponding data matrix:

$$K = \begin{bmatrix} C & B \\ A & D \end{bmatrix}.$$

Well-known:

$$X \text{ is a solution}$$
$$\Longleftrightarrow$$
$$\operatorname{range}\left(\begin{bmatrix} I \\ X \end{bmatrix}\right) \text{ is an } K\text{-invariant subspace corresponding to}$$

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Nonsymmetric Algebraic Riccati Equations

Here: nARE with zero Sylvester part (nAREzS)

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Of course, with X, also $Y = -X$ is a solution!

Recall: for (G, H)-polar decomposition, need invertible solution!
Trivial consequence: $\mathrm{rank}\,(A) = \mathrm{rank}\,(B)$ is necessary condition.

Theorem

Let $A, B \in \mathbb{C}^{n\times n}$. Then there exists a nonsingular matrix $X \in \mathbb{C}^{n\times n}$ solution of the nAREzS

$$0 = A - XBX$$

$\Longleftrightarrow$ there exists a matrix square root $M \in \mathbb{C}^{n\times n}$ of BA with

$$\ker A = \ker M \quad \text{and} \quad \ker B^* = \ker M^*.$$

Proof:
[$\Rightarrow$] Let X be a nonsingular solution. For $M = X^{-1}A = BX$: $BA = M^2$ as well as $\ker A = \ker M$. Since $X^*B^* = M^*$, we also have $\ker B^* = \ker M^*$.
[$\Leftarrow$] if $\mathrm{rank}\,(A) = \mathrm{rank}\,(B) = n$, then $X = AM^{-1}$ is a solution. Otherwise, construct suitable generalized inverse of M.

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Numerical Solution of nAREzS

Many possibilities:

use explicit solution $X = AM^{-1}$ (matrix square root of BA can be computed without forming product BA [B./Fassbender 2001]),

or special versions of

- Schur vector method [Laub 1979],
- Newton's method [Demmel 1987],
- sign function method,
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Compute Schur decomposition

$$\begin{bmatrix} & B \\ A & \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ V_1 & V_2 \end{bmatrix} = \begin{bmatrix} U_1 & U_2 \\ V_1 & V_2 \end{bmatrix} \begin{bmatrix} T_1 & S \\ & T_2 \end{bmatrix}.$$

Then: if U_1 is invertible, then

$$X = \pm V_1 U_1^{-1}$$

are solutions to $0 = A - XBX$.

Open questions:

- Under which conditions is U_1 nonsingular?
- Under which conditions is X nonsingular? (Obviously, if $\operatorname{rank}(V_1) = n$, but ...)
- How to exploit zero blocks in K?

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For $\mathcal{R}(X) = A - XBX$, Newton-Kantorovich method

$$\mathcal{R}'_{X_j}(Z_j) = -\mathcal{R}(X_j), \quad X_{j+1} = X_j + Z_j,$$

can be written as

- Solve Sylvester equation $(X_j B)Z_j + Z_j(BX_j) = A - X_j BX_j$.
- Set $X_{j+1} = X_j + Z_j$.

Conjecture: convergence from $X_0 = I_n$ for $\Lambda(AB) \cap \mathbb{R}_0^- = \emptyset$.

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Definition

For $Z \in \mathbb{R}^{n \times n}$ with $\Lambda(Z) \cap \imath\mathbb{R} = \emptyset$ and Jordan canonical form

$$Z = S^{-1} \begin{bmatrix} J^+ & 0 \\ 0 & J^- \end{bmatrix} S$$

the matrix sign function is

$$\operatorname{sign}(Z) := S \begin{bmatrix} I_k & 0 \\ 0 & -I_{n-k} \end{bmatrix} S^{-1}.$$

Computation of $\operatorname{sign}(Z)$

$\operatorname{sign}(Z)$ is root of $I_n \Longrightarrow$ use Newton's method to compute it:

$$Z_0 \leftarrow Z, \qquad Z_{j+1} \leftarrow \frac{1}{2}\left(Z_j + Z_j^{-1}\right), \qquad j = 1, 2, \ldots$$

$\Longrightarrow \quad \operatorname{sign}(Z) = \lim_{j\to\infty} Z_j.$

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Application to $K = \begin{bmatrix} & B \\ A & \end{bmatrix}$ yields for $A_0 = A$, $B_0 = B$:

$$A_{j+1} \leftarrow \frac{1}{2}\left(A_j + B_j^{-1}\right), \quad B_{j+1} \leftarrow \frac{1}{2}\left(B_j + A_j^{-1}\right),$$

for $j = 1, 2, \ldots$ and $X = \lim_{j\to\infty} A_j$ if $\Lambda(AB) \cap \mathbb{R}_0^- = \emptyset$.

Summary

- (G, H)-polar decompositions can be used to solve generalized Procrustes problems in non-Euclidian geometries — useful in psychometrics/multidimensional scaling.
- Construction of (G, H)-polar decompositions leads to nonsymmetric algebraic Riccati equations with zero Sylvester (linear) part.
- Efficient numerical algorithms for nAREzS not yet fully developed — work in progress!

Thanks for your attention!