MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Introduction to Model Reduction for Linear Dynamical Systems

Peter Benner

Compact Course "Scientific Computing"
Magdeburg, July 16–20, 2018

OTTO VON GUERICKE UNIVERSITÄT MAGDEBURG

DFG-Graduiertenkolleg
MATHEMATISCHE KOMPLEXITÄTSREDUKTION

http://www.mpi-magdeburg.mpg.de/mpcsc/benner/talks/lecture-MOR-MathCoRe.pdf

Outline

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Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

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This is the task of model reduction (also: dimension reduction, order reduction).

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input $=$ output of plant,
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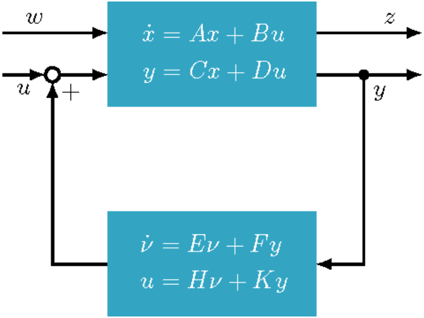
Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



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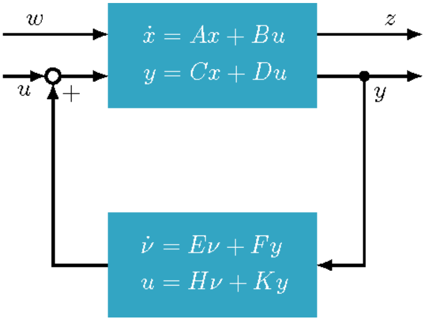

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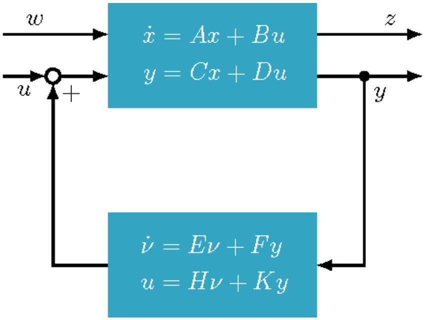

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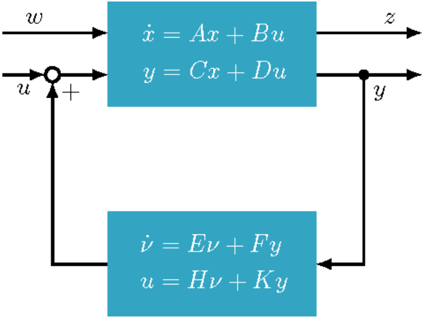

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Standard MOR techniques in systems and control: balanced truncation and related methods.

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Application Areas

Micro Electronics/Circuit Simulation

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- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines (interconnect, powergrid), parasitic effects,
  - modeling pin packages in VLSI chips,
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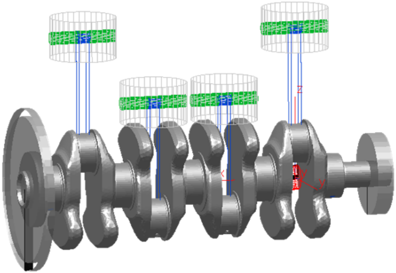
Standard MOR techniques in circuit simulation: Krylov subspace / Padé approximation / rational interpolation methods.

Application Areas
Structural Mechanics / Finite Element Modeling



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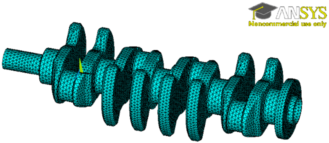

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- Analysis of elastic deformations requires many simulation runs for varying external forces.

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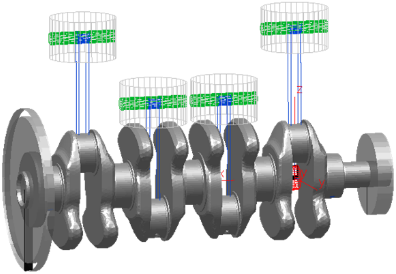

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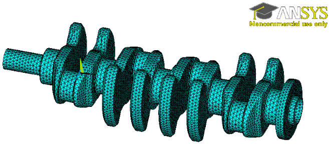

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Standard MOR techniques in structural mechanics: modal truncation, combined with Guyan reduction (static condensation) $\rightsquigarrow$ Craig-Bampton method.

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

An Inspiration: Image Compression by Truncated SVD

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Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
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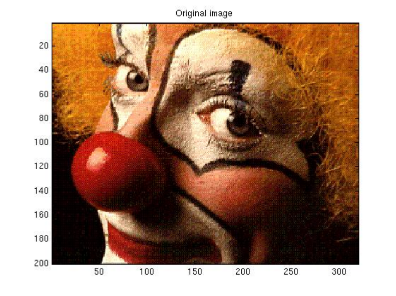
Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r$, $\sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kb

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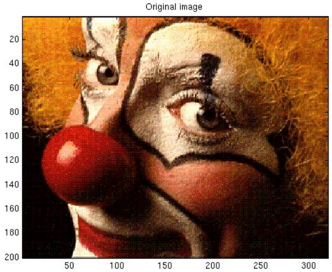

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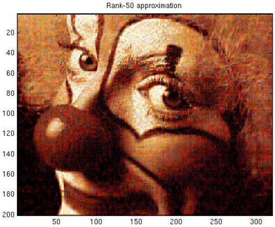
- rank $r = 50$, ≈ 104 kb



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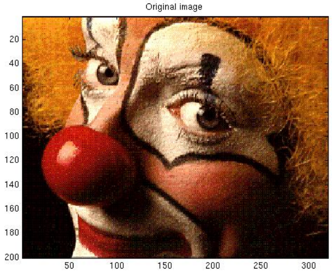

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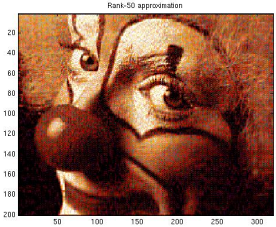
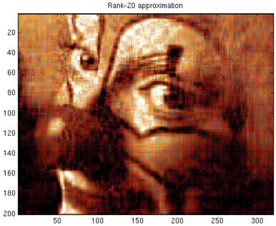
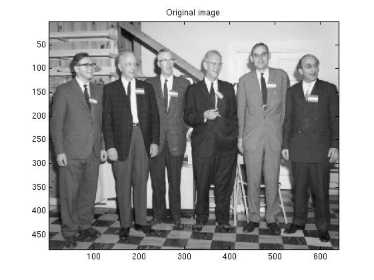


- rank $r = 20$, ≈ 42 kb



Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:
James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.



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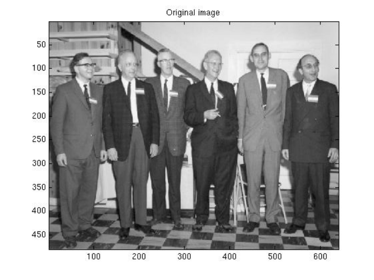
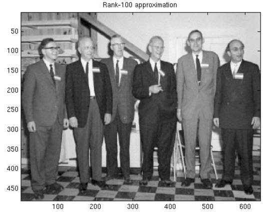


640 × 480 pixel, ≈ 1229 kb

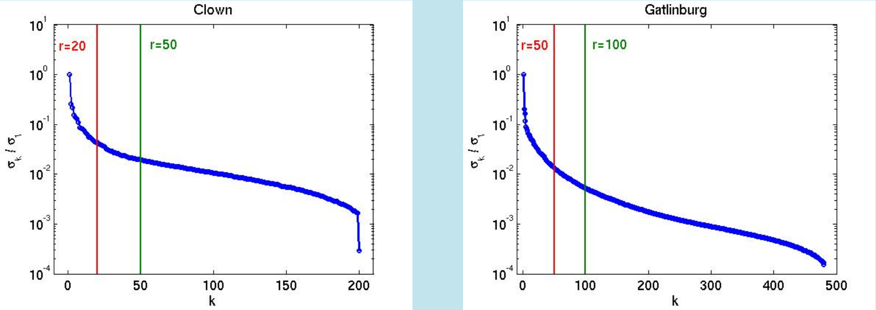
- rank $r = 100$, ≈ 448 kb



- rank $r = 50$, ≈ 224 kb

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices



Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) & = & f(t, x(t), u(t)), \quad x(t_0) = x_0, \\ y(t) & = & g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
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Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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Model Reduction for Dynamical Systems

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Secondary goal: reconstruct approximation of x from $\hat{x}$.

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}
\dot{x} &= f(t,x,u) = Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\
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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

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Variation-of-constants $\Longrightarrow$

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- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Alternative to State-Space Operator: Hankel operator

Instead of

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use Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau)\, d\tau \quad \text{for all } t > 0.$$

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$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \ldots \geq \sigma_n \geq \sigma_{n+1} = 0 = \ldots = 0.$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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$\Downarrow$

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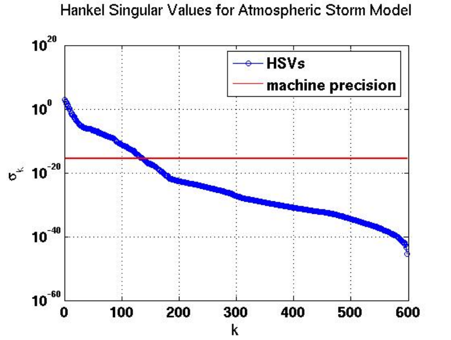
$\Downarrow$

Hankel singular values



Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx, \qquad\qquad C \in \mathbb{R}^{p\times n}. \end{aligned}$$

Alternative to State-Space Operator: Hankel operator

$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

$\mathcal{H}$ compact $\Rightarrow$ $\mathcal{H}$ has discrete SVD

$\Rightarrow$ Best approximation problem w.r.t. 2-induced operator norm well-posed

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But: computationally infeasible for large-scale systems.

Linear Systems in Frequency Domain

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Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

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Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)} \Big) u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty, \infty)))$.

Approximation Problem

Approximate the dynamical system

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

by reduced-order system

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}. \end{aligned}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| \leq \text{tolerance} \cdot \|u\|.$$

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$\Longrightarrow$ Approximation problem: $\min_{\text{order}\,(\hat{G}) \le r} \|G - \hat{G}\|$.

Definition

A linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is stable if its transfer function $G(s)$ has all its poles in the left half plane and it is asymptotically (or Lyapunov or exponentially) stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \mid \Re(z) < 0\}$.

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Lemma

Sufficient for asymptotic stability is that A is asymptotically stable (or Hurwitz), i.e., the spectrum of A, denoted by $\Lambda(A)$, satisfies $\Lambda(A) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

Questions:

- For fixed $x_0 \in \mathbb{R}^n$ and some $x^1 \in \mathbb{R}^n$, is there a feasible control function $u \in \mathcal{U}_{ad}$ (e.g., $\mathcal{U}_{ad} \in \{C^k[0,T], L_2(0,T), PC[0,T]\}$, possibly with constraints $\underline{u}(t) \leq u(t) \leq \overline{u}(t)$) and time $t_1 > t_0 = 0$ such that $x(t_1; u) = x^1$?
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Note: for LTI systems $\dot{x} = Ax + Bu$, both concepts are equivalent!

Definition (Controllability)

Consider the target (the state to be reached) $x^1 \in \mathbb{R}^n$.

a) An LTI system with initial value $x(0) = x^0$ is controllable to x^1 in time $t_1 > 0$ if there exists $u \in \mathcal{U}_{ad}$ such that $x(t_1; u) = x^1$.

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The controllability set w.r.t. x^1 is defined as $\mathcal{C} := \bigcup_{t_1 > 0} \mathcal{C}(t_1)$ where

$$\mathcal{C}(t_1) := \{x^0 \in \mathbb{R}^n \mid \exists u \in \mathcal{U}_{ad} : x(t_1; u) = x^1\}.$$

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In short: an LTI system is controllable $\iff$ $\mathcal{C} = \mathbb{R}^n$.

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Variation of constants $\Longrightarrow$

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds = e^{At}(x^0 + \int_0^t e^{-As}Bu(s)ds).$$

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Ansatz: $u(t) = B^T e^{-A^T t}c \Longrightarrow$

$$e^{-At_1}x^1 - x^0 = \int_0^{t_1} e^{-At}BB^T e^{-A^T t}dt\, c =: P(0, t_1)c.$$

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Hence, an LTI system is controllable iff this linear system is solvable for $c \in \mathbb{R}^n$, i.e., iff $P(0, t_1)$ is invertible. (Note: $P(0, t_1) = P(0, t_1)^T \geq 0$ by definition!)

Now: characterize controllability.

Theorem

For an LTI system defined by $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, *T.F.A.E.:*

a) *The LTI system* $\dot{x} = Ax + Bu$ *is controllable.*

b) *The finite time Gramian* $P(0, t_1)$ *is spd* $\forall\ t_1 > 0$.

c) *The controllability matrix*

$$K(A, B) := [B, AB, A^2B, \ldots, A^{n-1}B] \in \mathbb{R}^{n\times n\cdot m}$$

has full rank n. (Note: $\mathrm{range}\,(K(A, B)) = \mathcal{C}(t_1)\ \forall\ t_1 > 0$*!)*

d) *If z is a left eigenvector of A, then* $z^*B \neq 0$.

e) *(Hautus test)* $\mathrm{rank}([\lambda I - A, B]) = n\ \forall\ \lambda \in \mathbb{C}$.

The Gramian characterization of controllability for stable systems can be based on positive definiteness of the (infinite) controllability Gramian

$$P := \int_0^\infty e^{As} B B^T e^{A^T s} ds,$$

using congruence of $P(0, t_1)$ to $\int\limits_0^{t_1} e^{As} B B^T e^{A^T s} ds$ and taking the limit $t_1 \to \infty$.

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Theorem

For a stable LTI system defined by $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$, T.F.A.E.:

a) *The LTI system $\dot{x} = Ax + Bu$ is controllable.*

b) *The controllability Gramian P is positive definite.*

New question: suppose we have

$$y(t) = \tilde{y}(t)$$

corresponding to two trajectories $x, \tilde{x}$ obtained by the same input function $u(t)$. Can we conclude that $x(0) = \tilde{x}(0)$, or even stronger, that $x(t) = \tilde{x}(t)$ for $t \leq 0, t \geq 0$ (past/future)?

(Note that $x(t_0) = \tilde{x}(t_0)$ is sufficient as trajectory uniquely determined. In other words, is the mapping $x^0 \to y(t)$ injective?)

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Definition (Observability)

An LTI system is reconstructable (observable) if for solution trajectories $x(t), \tilde{x}(t)$ obtained with the same input function u, we have

$$\begin{aligned} y(t) = \quad & \tilde{y}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0) \\ \Longrightarrow \quad x(t) = \quad & \tilde{x}(t) \quad \forall t \leq 0 \quad (\forall t \geq 0). \end{aligned}$$

Characterization of observability/reconstructability:

Theorem (Duality)

An LTI system is reconstructable if and only if the dual system
$\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

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For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, T.F.A.E.:

a) *The LTI system is reconstructable.*

b) *The LTI system is observable.*

c) *The observability matrix*

$$\mathcal{O}(A, C) = \left[C^T, A^T C^T, (A^2)^T C, \ldots, (A^{n-1})^T C^T\right]^T \in \mathbb{R}^{np\times n} \text{ has rank } n.$$

d) *If $Ax = \lambda x$, then $C^T x \neq 0$.*

e) *(Hautus test)* $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n.$

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An LTI system is reconstructable if and only if the dual system
$\dot{x}(t) = -A^T x(t) - C^T u(t)$ *is controllable.*

Theorem

A stable LTI system is observable if and only if the observability Gramian

$$Q := \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is symmetric positive definite.

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Theorem

For an LTI system defined by $(A,B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$, T.F.A.E.:

a) *The LTI system is stabilizable.*

b) $\exists F \in \mathbb{R}^{m\times n}$ *with* $\Lambda(A+BF) \subset \mathbb{C}^-$.

c) *If* $p^*A = \tilde{\lambda} p^*$ *and* $\mathrm{Re}(\lambda) \geq 0$, *then* $p^*B \neq 0$.

d) $\mathrm{rank}([A - \lambda I, B]) = n \quad \forall \lambda \in \mathbb{C}$ *with* $\mathrm{Re}(\lambda) \geq 0$.

e) $\Lambda(A_3) \subset \mathbb{C}^-$ *in the (controllability) Kalman decomposition of* (A,B),

$$V^T A V = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}, V^T B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

$\exists$ dual concept of stabilizability, analogous to duality of controllability and observability.

Definition (Detectability)

An LTI system is detectable if for any solution $x(t)$ of $\dot{x} = Ax$ with $Cx(t) \equiv 0$ we have $\lim_{t\to\infty} x(t) = 0$.
(We can not observe all of x, but the unobservable part is stable.)

∃ dual concept of stabilizability, analogous to duality of controllability and observability.

Theorem

For an LTI system defined by $(A, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{p\times n}$, *T.F.A.E.:*

a) *The LTI system is detectable.*

b) (A^T, C^T) *is stabilizable.*

c) $Ax = \lambda x, \mathrm{Re}(\lambda) \geq 0 \Rightarrow C^T x \neq 0.$

d) $\mathrm{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ *for all* $\lambda, \mathrm{Re}(\lambda) \geq 0.$

e) *In the observability Kalman decomposition of* (A^T, C^T),

$$W^T A W = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix}, CW = [C_1 \; 0],$$

we have $\Lambda(A_3) \subset \mathbb{C}^-$.

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & \text{with transfer function} \\ y(t) = Cx(t) + Du(t), & G(s) = C(sI - A)^{-1}B + D, \end{cases}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

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Realizations are not unique!

Transfer function is invariant under state-space transformations,

$$\mathcal{T} : \begin{cases} x & \to & Tx, \\ (A, B, C, D) & \to & (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

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For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{cases} \quad \begin{array}{l} \text{with transfer function} \\ G(s) = C(sI - A)^{-1}B + D, \end{array}$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \times \mathbb{R}^{p \times m}$ is called a realization of Σ.

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Definition

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Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

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Definition

The McMillan degree of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

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A minimal realization is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

Theorem

A realization (A, B, C, D) of a linear system is minimal $\Longleftrightarrow$
(A, B) is controllable and (A, C) is observable.

Definition

A realization (A, B, C, D) of a linear system Σ is balanced if its infinite controllability/observability Gramians P/Q satisfy

$$P = Q = \mathrm{diag}\{\sigma_1, \ldots, \sigma_n\} \quad \text{(w.l.o.g. } \sigma_j \geq \sigma_{j+1},\ j = 1, \ldots, n-1).$$

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When does a balanced realization exist?
Assume A to be Hurwitz, i.e. $\Lambda(A) \subset \mathbb{C}^-$. Then:

Theorem

Given a stable minimal linear system $\Sigma : (A, B, C, D)$, a balanced realization is obtained by the state-space transformation with

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$, $Q = R^T R$ (e.g., Cholesky decompositions) and $SR^T = U \Sigma V^T$ is the SVD of SR^T.

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Note: $\sigma_1, \ldots, \sigma_n \geq 0$ as $P, Q \geq 0$ by definition, and $\sigma_1, \ldots, \sigma_n > 0$ in case of minimality!

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$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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Proof. Exercise!

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Theorem

The Hankel singular values (HSVs) of a stable minimal linear system are system invariants, i.e. they are unaltered by state-space transformations!

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Proof. In balanced coordinates, the HSVs are $\Lambda\left(PQ\right)^{\frac{1}{2}}$. Now let

$$(\hat{A}, \hat{B}, \hat{C}, D) = (TAT^{-1}, TB, CT^{-1}, D)$$

be any transformed realization with associated controllability Lyapunov equation

$$0 = \hat{A}\hat{P} + \hat{P}\hat{A}^T + \hat{B}\hat{B}^T = TAT^{-1}\hat{P} + \hat{P}T^{-T}A^TT^T + TBB^TT^T.$$

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The uniqueness of the solution of the Lyapunov equation implies that $\hat{P} = TPT^T$ and, analogously, $\hat{Q} = T^{-T}QT^{-1}$. Therefore,

$$\hat{P}\hat{Q} = TPQT^{-1},$$

showing that $\Lambda\left(\hat{P}\hat{Q}\right) = \Lambda\left(PQ\right) = \{\sigma_1^2, \ldots, \sigma_n^2\}$.

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Remark

For non-minimal systems, the Gramians can also be transformed into diagonal matrices with the leading $\hat{n} \times \hat{n}$ submatrices equal to $\operatorname{diag}(\sigma_1, \ldots, \sigma_{\hat{n}})$, and

$$\hat{P}\hat{Q} = \operatorname{diag}(\sigma_1^2, \ldots, \sigma_{\hat{n}}^2, 0, \ldots, 0).$$

see [Laub/Heath/Paige/Ward 1987, Tombs/Postlethwaite 1987].

Consider the transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

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(Here:, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

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$$\Longrightarrow y \in L_2^p(-\infty, \infty) \cong \mathcal{L}_2^p.$$

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Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty := \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\,(G(\jmath\omega))\,.$$

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Hardy space $\mathcal{H}_\infty$

Function space of analytic and bounded (in $\mathbb{C}^+$) matrix-/scalar-valued functions. The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\mathrm{Re}\,(s)>0} \sigma_{\max}(F(s)) = \sup_{\omega\in\mathbb{R}} \sigma_{max}(F(\jmath\omega)).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

Consider the transfer function

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \le \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic in $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\|F\|_2 := \left(\sup_{\mathrm{Re}\,(\sigma)>0} \int_{-\infty}^{\infty} \|F(\sigma + \jmath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}$$
$$= \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2 \, d\omega \right)^{\frac{1}{2}}.$$

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$$\|F\|_2 \;=\; \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F^2\, d\omega\right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$ approximation error for impulse response ($u(t) = u_0\delta(t)$)

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \le \|G - \hat{G}\|_2\|u_0\|.$$

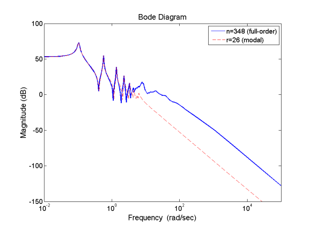
$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

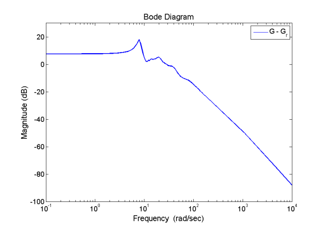
$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory)

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot:
  - for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}(\text{value})$;
  - for MIMO systems, $p \times m$ array of of plots G_{ij}.





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- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

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  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

(“system does not generate energy”).

Projector

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

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- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\,w_1, \ldots, w_r\,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^TV)^{-1}W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

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Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\operatorname{range}(V) = \mathcal{V}, \quad \operatorname{range}(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

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$$\hat{A} := W^T A V, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observation:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T \left(VW^T\dot{x} - AVW^T x - Bu\right)$$

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Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^Tx =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^Tx$

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Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

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$P(s)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s)) \subset \mathrm{range}\,(V)$, all matrices have full rank ⇒ "=", and

$$P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)$$

Model Reduction by Projection

Projection $\rightsquigarrow$ Rational Interpolation

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$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$\begin{aligned} G(s) - \hat{G}(s) &= \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right) \\ &= C\left(I_n - \underbrace{V(sI_r - \hat{A})^{-1}W^T(sI_n - A)}_{=:P(s)}\right)(sI_n - A)^{-1}B. \end{aligned}$$

$P(s)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

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$$\text{Analogously,} \quad = \quad C(sI_n - A)^{-1}\left(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\right)B.$$

$Q(s)^*$ is a projector onto $\mathcal{W}$ $\Longrightarrow$ Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$

hence $G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*)$, i.e., $\hat{G}$ interpolates G in s_*!

Theorem [Grimme 1997, Villemagne/Skelton 1987]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \mathrm{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \mathrm{range}(W)$,

then at $s = s_*$, we obtain the (rational) interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

Note: extension to Hermite interpolation conditions later!

Outline

Modal Truncation

Basic method:

Assume A is diagonalizable: $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresponding to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W}^* = T^{-1}(1:r, :), \quad W = \tilde{W}(V^*\tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^*AV = \mathrm{diag}\,\{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^*B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \ C_2 \,], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \le \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\,\hat{C},\ C_2\,] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2 \end{aligned}$$

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Proof:

$$G(s) \quad = \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2\,\mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \ldots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

Modal Truncation

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Assume A is diagonalizable: $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, t_k = eigenvectors corresponding to "dominant" modes / eigenvalues of A. Then reduced-order model is

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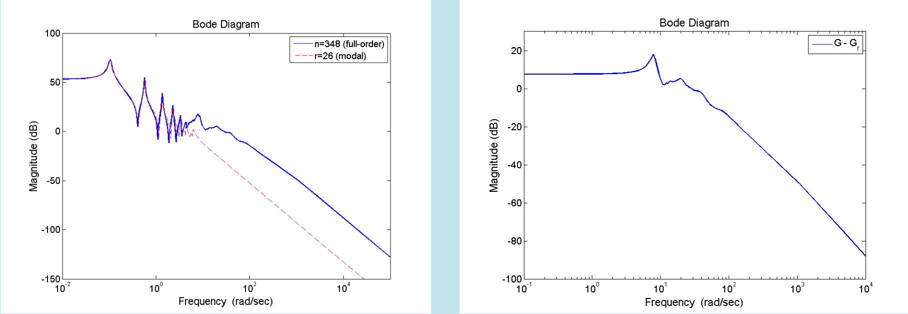
Difficulties:

- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([Litz '79] use Jordan canoncial form; otherwise merely heuristic criteria, e.g., [Varga '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = p = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



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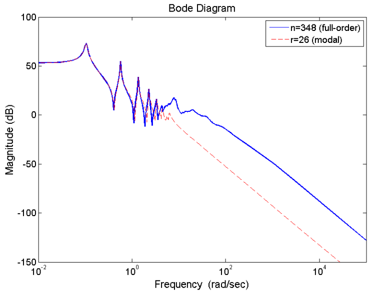

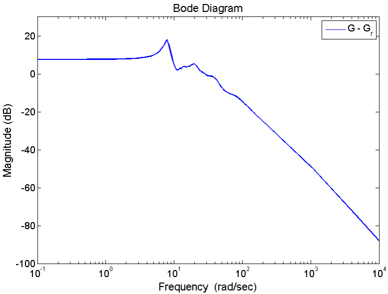

MATLAB® demo.

Base enrichment

Static modes are defined by setting $\dot{x} = 0$ and assuming unit loads, i.e., $u(t) \equiv e_j$, $j = 1, \ldots, m$:

$$0 = Ax(t) + Be_j \quad \Longrightarrow \quad x(t) \equiv -A^{-1} b_j.$$

Projection subspace $\mathcal{V}$ is then augmented by $A^{-1}[\, b_1, \ldots, b_m \,] = A^{-1}B$.

Interpolation-projection framework $\Longrightarrow G(0) = \hat{G}(0)$!

If two sided projection is used, complimentary subspace can be augmented by $A^{-T}C^T \Longrightarrow G'(0) = \hat{G}'(0)$!

Note: if $m \neq q$, add random vectors or delete some of the columns in $A^{-T}C^T$.

Guyan reduction (static condensation)

Partition states in masters $x_1 \in \mathbb{R}^r$ and slaves $x_2 \in \mathbb{R}^{n-r}$ (FEM terminology)
Assume stationarity, i.e., $\dot{x} = 0$ and solve for x_2 in

$$\begin{aligned} 0 &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \\ \Rightarrow \quad x_2 &= -A_{22}^{-1} A_{21} x_1 - A_{22}^{-1} B_2 u. \end{aligned}$$

Inserting this into the first part of the dynamic system

$$\dot{x}_1 = A_{11} x_1 + A_{12} x_2 + B_1 u, \quad y = C_1 x_1 + C_2 x_2$$

then yields the reduced-order model

$$\begin{aligned} \dot{x}_1 &= (A_{11} - A_{12} A_{22}^{-1} A_{21}) x_1 + (B_1 - A_{12} A_{22}^{-1} B_2) u \\ y &= (C_1 - C_2 A_{22}^{-1} A_{21}) x_1 - C_2 A_{22}^{-1} B_2 u. \end{aligned}$$

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{p \times m}$.

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Note: this follows using the spectral decomposition $A = XDX^{-1}$, with $X = [x_1, \ldots, x_n]$ the right and $X^{-1} =: Y = [y_1, \ldots, y_n]^H$ the left eigenvector matrices:

$$\begin{aligned} G(s) &= C(sI - XDX^{-1})^{-1}B = CX(sI - \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\})^{-1}YB \\ &= [\, Cx_1, \ldots, Cx_n \,] \begin{bmatrix} \frac{1}{s-\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} y_1^H B \\ \vdots \\ y_n^H B \end{bmatrix} \\ &= \sum_{k=1}^{n} \frac{(Cx_k)(y_k^H B)}{s - \lambda_k}. \end{aligned}$$

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with the residues $R_k := (Cx_k)(y_k^H B) \in \mathbb{C}^{p \times m}$.

Note: $R_k = (Cx_k)(y_k^H B)$ are the residues of G in the sense of the residue theorem of complex analysis:

$$\mathrm{res}\,(G, \lambda_\ell) = \lim_{s \to \lambda_\ell} (s - \lambda_\ell) G(s) = \sum_{k=1}^{n} \underbrace{\lim_{s \to \lambda_\ell} \frac{s - \lambda_\ell}{s - \lambda_k}}_{= \begin{cases} 0 \text{ for } k \neq \ell \\ 1 \text{ for } k = \ell \end{cases}} R_k = R_\ell.$$

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function with $D = 0$:

$$G(s) = \sum_{k=1}^{n} \frac{R_k}{s - \lambda_k}$$

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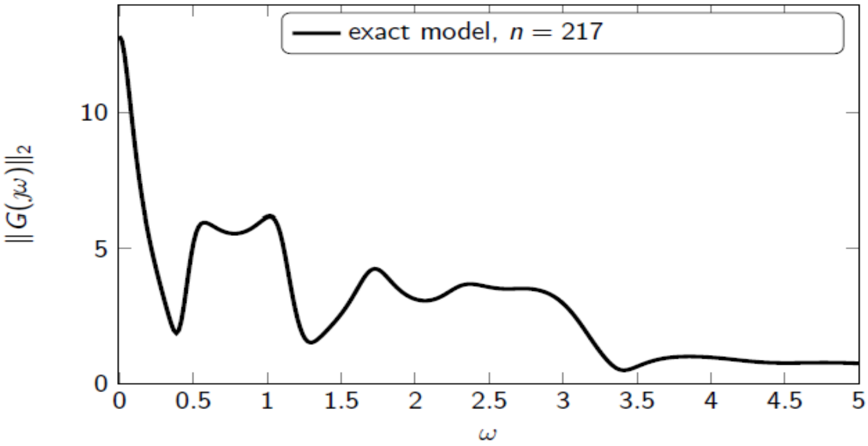
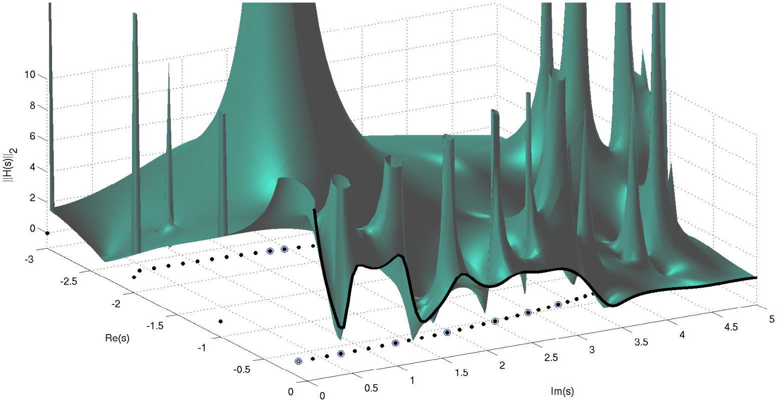
Remark

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks"' in the frequency response.

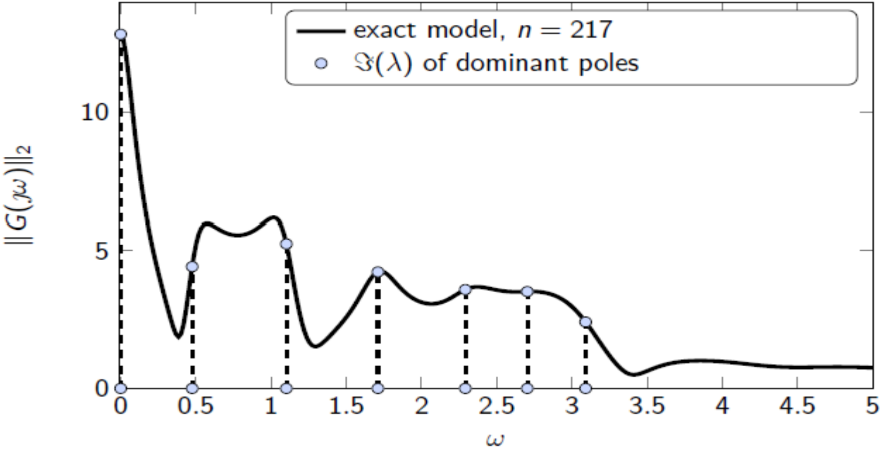
Frequency response / Bode (magitude) plot of transfer function



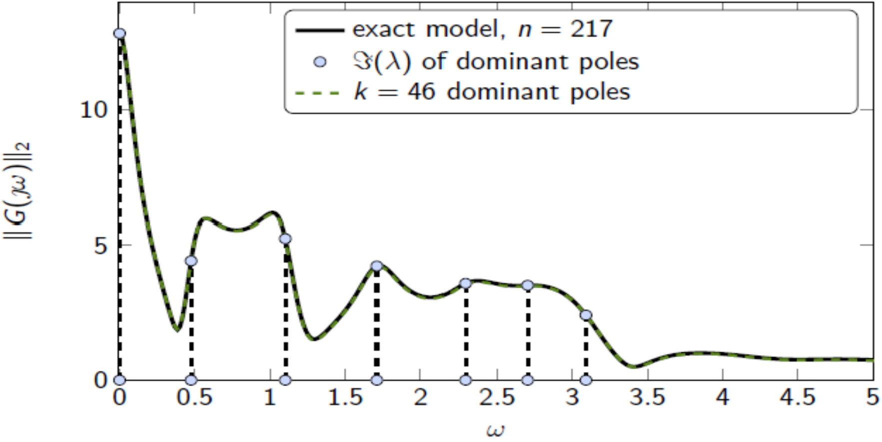
Transfer function in the upper left quadrant



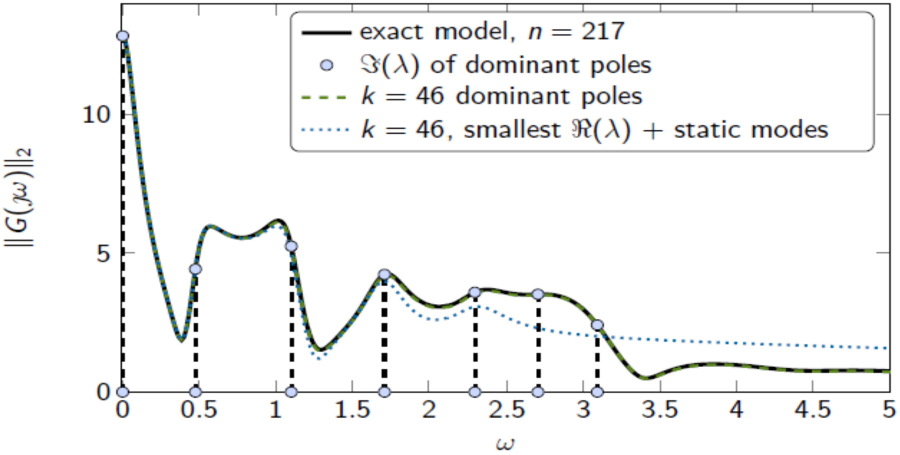
Dominant poles



Reduced-order model using dominant poles



Reduced-order model using poles w/ min real parts + static modes



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$\Longleftrightarrow \qquad PQz = \sigma^2 z. \quad \square$

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- Compute balanced realization of the system via state-space transformation

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- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$:

in transformed coordinates, the Gramians satisfy

$$
\begin{aligned}
(TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\
(TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0
\end{aligned}
$$

$$
\Rightarrow \; (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},
$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

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HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates ... energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\, dt}{\int_{-\infty}^0 u(t)^T u(t)\, dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs

$\Longrightarrow$ complete analogy to best approximation via SVD!

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2. Compute SVD $SR^T = [\, U_1,\ U_2\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\, I_r,\ 0\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} \end{aligned}$$

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$\Longrightarrow$ VW^T is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.

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- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

Properties:

General misconception: complexity $\mathcal{O}(n^3)$ – true for several implementations! (e.g., MATLAB, SLICOT).

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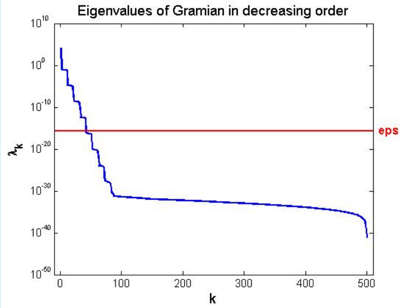
Use low-rank techniques ideas from numerical linear algebra:

- Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

- Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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Use low-rank techniques ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Implementation using sparse Lyapunov solver ($\rightarrow$ADI$+$sparse LU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - $+$ MATLAB toolbox LyaPack (Penzl 1999),
  - $+$ Software library M.E.S.S.[a] in C/MATLAB [B./Saak/Köhler].

[a] Matrix Equation Sparse Solvers

Recall Peaceman-Rachford ADI:

Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$.

ADI iteration idea: decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$\begin{aligned} (H + pI)v &= r \\ (V + pI)w &= t \end{aligned}$$

can be solved easily/efficiently.

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ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots,$ such that

$$\begin{array}{rcl} u_0 &=& 0\\ (H + p_k I)u_{k-\frac{1}{2}} &=& (p_k I - V)u_{k-1} + s\\ (V + p_k I)u_k &=& (p_k I - H)u_{k-\frac{1}{2}} + s \end{array}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

ADI Methods for Lyapunov Equations

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation **[Wachspress 1988]**

$$\begin{aligned} X_0 &= 0, \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I), \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n\times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation **[Wachspress 1988]**

For $k = 1, \ldots, k_{\max}$

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T(A^T - p_k I) \end{aligned}$$

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Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\ &\quad +(A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T} \end{aligned}$$

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... $\rightsquigarrow$ low-rank Cholesky factor ADI [Penzl 1997/2000, Li/White 1999/2002, B./Li/Penzl 1999/2008, Gugercin/Sorensen/Antoulas 2003]

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B,\ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$ [Penzl '00]

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Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}},\ P_{k_{\max}-1}z_{k_{\max}},\ P_{k_{\max}-2}(P_{k_{\max}-1}z_{k_{\max}}),\ \ldots,\ P_1(P_2 \cdots P_{k_{\max}-1}z_{k_{\max}})],$$

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1}B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I - (p_i + p_{i+1})(A + p_i I)^{-1}\right].$$

[Li/White '02]

$$Z_k = [\sqrt{-2p_k}(A+p_kI)^{-1}B,\ (A+p_kI)^{-1}(A-p_kI)Z_{k-1}]$$ [Penzl '00]

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$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}}\left[I-(p_i+p_{i+1})(A+p_iI)^{-1}\right].$$

[Li/White '02]

$\rightsquigarrow$ Need to solve only one (sparse) linear system with m right-hand sides per iteration!

Algorithm [Penzl 1997/2000, Li/White 1999/2002, B. 2004, B./Li/Penzl 1999/2008]

$$V_1 \leftarrow \sqrt{-2\mathrm{Re}\,(p_1)}(A + p_1 I)^{-1}B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{Re}\,(p_k)}{\mathrm{Re}\,(p_{k-1})}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Z_k \leftarrow \begin{bmatrix} Z_{k-1} & V_k \end{bmatrix}$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{\% column compression, optional}$$

Note: Implementation in real arithmetic possible by combining two steps [B./Li/Penzl 1999/2008] or employing the relation of 2 consecutive complex factors [B./Kürschner/Saak 2011].

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At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \Big[\Big] \in \mathbb{C}^{n\times m}.$$

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Numerical Results for ADI

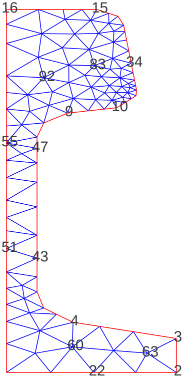
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, \qquad \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), \quad \xi \in \Gamma_k, \; 1 \le k \le 7, \\
\frac{\partial}{\partial n} x &= 0, \qquad \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$

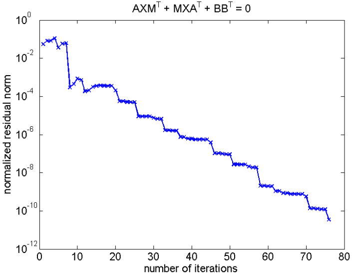


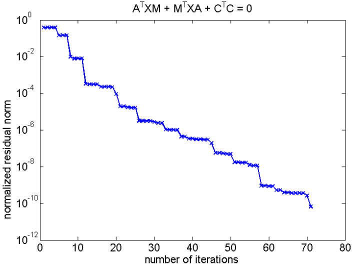
Source: Physical model: courtesy of Mannesmann/Demag.
Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \quad A^T QM + M^T QA + C^T C = 0,$$

for $79,841$.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- M.E.S.S. requires no factorization of mass matrix.
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.





Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\operatorname{range}(Z)$, $Z \in \mathbb{R}^{n \times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T A Z$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \operatorname{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].

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  [Saad 1990, Jaimoukha/Kasenally 1994, Jbilou 2002–08].
- Extended (and rational) Krylov method (EKSM, RKSM) [Simoncini 2007, Druskin/Knizhnerman/Simoncini 2011],

  $$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

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Examples:

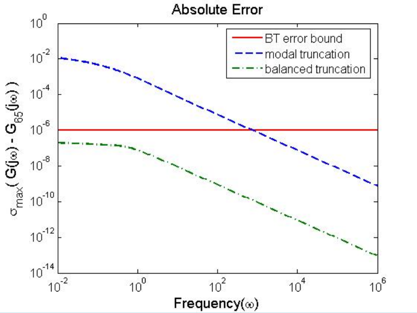
- ADI subspace [B./R.-C. Li/Truhar 2008]:

$$\mathcal{Z} = \mathrm{colspan}\left[\; V_1, \quad \ldots, \quad V_r \;\right].$$

Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White 2002].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou 2008].

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

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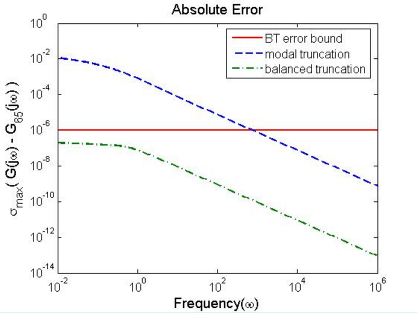
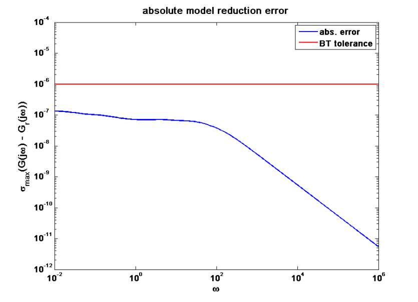


– BT model computed with sign function method,

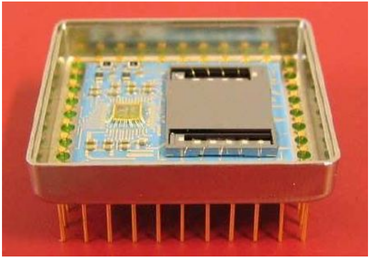
– MT w/o static condensation, same order as BT model.

$n = 79841$, Absolute Error

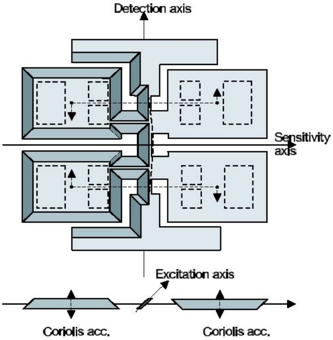


– BT model computed using M.E.S.S. in MATLAB,

– dualcore, computation time: <10 min.



- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: http://modelreduction.org/index.php/Modified_Gyroscope

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
  $\rightsquigarrow n = 34,722,\ m = 1,\ p = 12.$
- Reduced model computed using SpaRed, $r = 30$.

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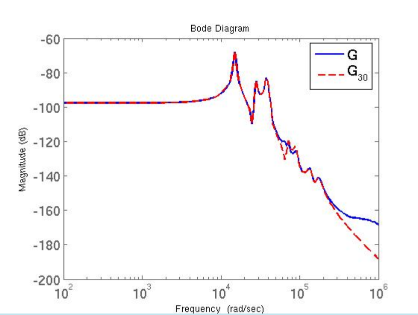
Frequency Repsonse Analysis



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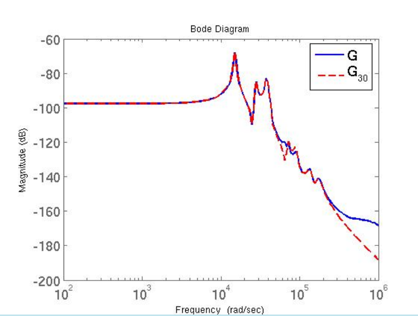
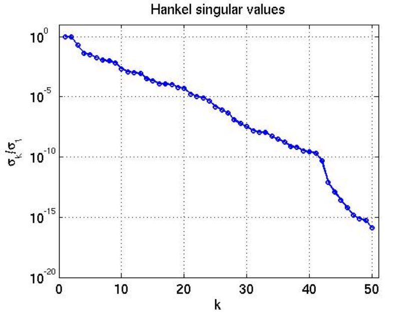


Hankel Singular Values



Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts 1976, Moore 1981]

- $P =$ controllability Gramian of system given by (A, B, C, D).
- $Q =$ observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0.$$

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LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman 1983]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

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$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST)

[Desai/Pal 1984, Green 1988]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^T Q + Q\bar{A} + QB\bar{R}^{-1}B^T Q + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

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\text{BT:} \quad & \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \sigma_j^{BT}, \\
\text{LQGBT:} \quad & \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}} \\
\text{BST:} \quad & \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,
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- Computations can be modularized $\rightsquigarrow$ software packages M-M.E.S.S., MORLAB, see http://www.mpi-magdeburg.mpg.de/823508/software.

Outline

Padé Approximation

Idea:

- Consider

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  - For $s_0 = 0$: $m_j := C(A^{-1})^j B =$ moments.
  - For $s_0 = \infty$: $m_j := CA^{j-1}B =$ Markov parameters.

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- As reduced-order model use rth Padé approximant $\hat{G}$ to G:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$
$\rightsquigarrow$ moment matching if $s_0 < \infty$,
$\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}\tilde{B}) =: \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0 I_n - A)^{-1}$, $\tilde{B} = (s_0 I_n - A)^{-1} B$) along

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- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.

Remark: Arnoldi (PRIMA) yields only $G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^r)$.

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- Preservation of physical properties only in special cases; usually requires post processing which (partially) destroys moment matching properties.

Computation of reduced-order model by projection

Given an LTI system $\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B,$ a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$ by computing

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

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Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i}G(s_j) = \frac{d^i}{ds^i}\hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Theorem (simplified) [Grimme 1997, Villemagne/Skelton 1987]

If

$$\operatorname{span}\left\{(s_1 I_n - A)^{-1}B, \ldots, (s_k I_n - A)^{-1}B\right\} \subset \operatorname{Ran}(V),$$
$$\operatorname{span}\left\{(s_1 I_n - A)^{-T}C^T, \ldots, (s_k I_n - A)^{-T}C^T\right\} \subset \operatorname{Ran}(W),$$

then

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Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

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Remarks:

$k = 1$, standard Krylov subspace(s) of dimension K $\rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1) = \frac{d^i}{ds^i}\hat{G}(s_1), \quad i = 0, \dots, K-1(+K).$$

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme 1997],
- Iterative Rational Krylov Algorithm (IRKA) [Antoulas/Beattie/Gugercin 2007].

Best $\mathcal{H}_2$-norm approximation problem

$$\text{Find} \quad \arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$$

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$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For SISO systems

$$G(-\mu_i) = \hat{G}(-\mu_i),$$
$$G'(-\mu_i) = \hat{G}'(-\mu_i),$$

where μ_i are the poles of the reduced transfer function $\hat{G}$.

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For MIMO systems

$$\begin{aligned} G(-\mu_i)\tilde{B}_i &= \hat{G}(-\mu_i)\tilde{B}_i, && \text{for } i = 1,\ldots,r,\\ \tilde{C}_i^T G(-\mu_i) &= \tilde{C}_i^T \hat{G}(-\mu_i), && \text{for } i = 1,\ldots,r,\\ \tilde{C}_i^T G'(-\mu_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\mu_i)\tilde{B}_i, && \text{for } i = 1,\ldots,r, \end{aligned}$$

where $T^{-1}\hat{A}T = \operatorname{diag}\{\mu_1,\ldots,\mu_r\}$ = spectral decomposition and

$$\tilde{B} = \hat{B}^T T^{-T}, \quad \tilde{C} = \hat{C}T.$$

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$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. 2006, Bunse-Gerstner et al. 2007, Van Dooren et al. 2008]

Algorithm 1 IRKA

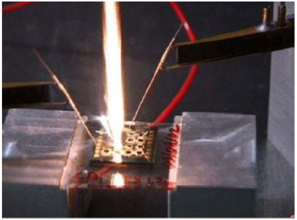
Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.

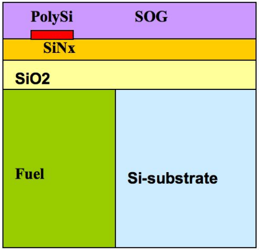
Output: A^{opt}, B^{opt}, C^{opt}

1: $\{\mu_1, \dots, \mu_r\} = \Lambda(\hat{A})$
2: **while** $(\max_{j=1,\dots,r} \left\{ |\mu_j - \mu_j^{\text{old}}| \,/\, |\mu_j| \right\} > \delta)$ **do**
3: $\quad \text{diag}\{\mu_1, \dots, \mu_r\} := T^{-1}\hat{A}T, \;\; \tilde{B} = \hat{B}^* T^{-*}, \;\; \tilde{C} = \hat{C}T.$
4: $\quad V = \left[(-\mu_1 I - A)^{-1} B\tilde{B}_1, \dots, (-\mu_r I - A)^{-1} B\tilde{B}_r \right]$
5: $\quad W = \left[(-\mu_1 I - A^T)^{-1} C^T\tilde{C}_1, \dots, (-\mu_r I - A^T)^{-1} C^T\tilde{C}_r \right]$
6: $\quad V = \text{orth}(V), \; W = \text{orth}(W)$
7: $\quad \hat{A} = (W^*V)^{-1} W^*AV, \; \hat{B} = (W^*V)^{-1} W^*B, \; \hat{C} = CV$
8: **end while**
9: $A^{opt} = \hat{A}, \; B^{opt} = \hat{B}, \; C^{opt} = \hat{C}$

Outline

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretization of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: http://modelreduction.org/index.php/Micropyros_Thruster

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow$ $n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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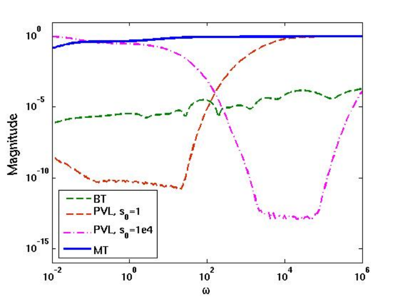

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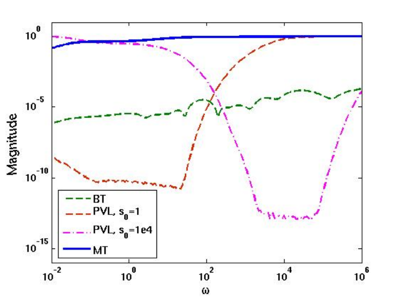
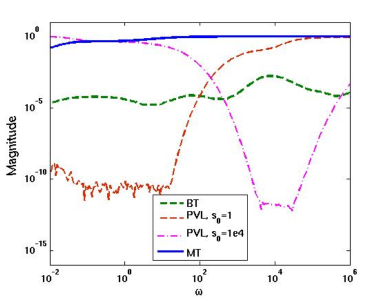


Relative error $n = 11,445$



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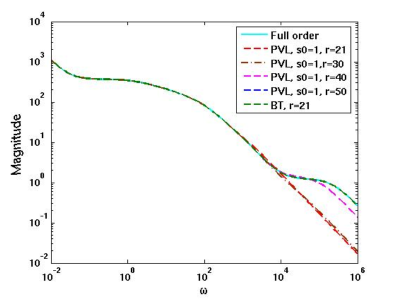
Frequency Response BT/PVL



Numerical Comparison of MOR Approaches

Microthruster

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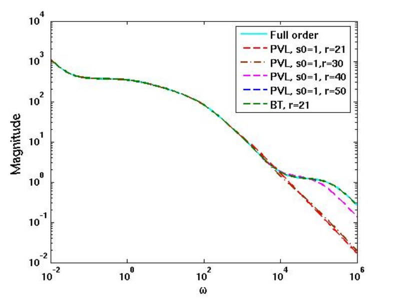
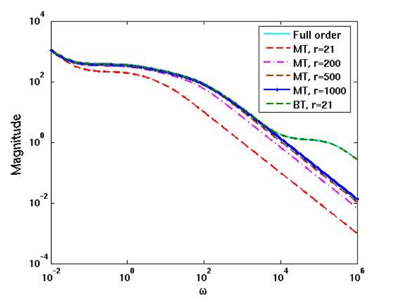


Frequency Response BT/MT



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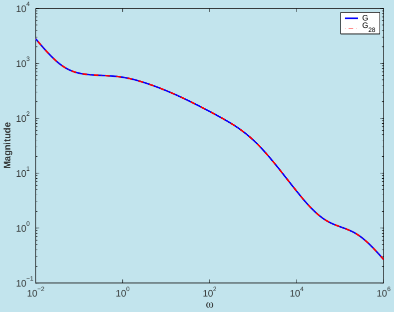
- Order of reduced model: $r = 28$.

Frequency Repsonse Analysis

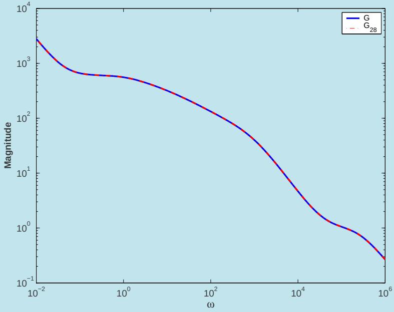


Numerical Comparison of MOR Approaches

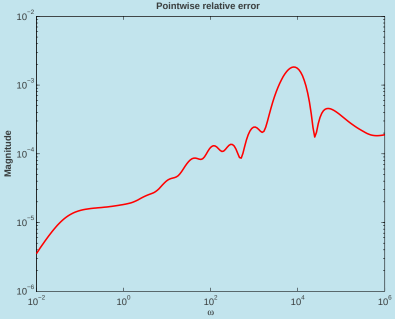
Microthruster

- axial-symmetric 2D model
- FEM discretization using quadratic elements $\rightsquigarrow$ $n = 11,445$, $m = 1$, $p = 7$.
- Reduced model computed with LyaPack [*Penzl 1999*].
- Order of reduced model: $r = 28$.

Frequency Repsonse Analysis



Relative Error



- Special methods for second-order (mechanical) and delay systems.
- Extensions to bilinear, quadratic-bilinear, polynomial, and stochastic systems.
- Rational interpolation methods for nonlinear systems.
- Other MOR techniques like proper orthogonal decomposition (POD) or the reduced basis method (RBM).
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

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