Summer School on
Numerical Linear Algebra
for Dynamical and High-Dimensional Problems
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Model Reduction for Linear Dynamical Systems

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Computational Methods in Systems and Control Theory
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http://www.mpi-magdeburg.mpg.de/mpcsc/benner/talks/lecture-MOR.pdf

Outline

Introduction

Model Reduction — Abstract Definition

Problem

Given a physical problem with dynamics described by the states $x \in \mathbb{R}^n$, where n is the dimension of the state space.

Because of redundancies, complexity, etc., we want to describe the dynamics of the system using a reduced number of states.

This is the task of model reduction (also: dimension reduction, order reduction).

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Application Areas

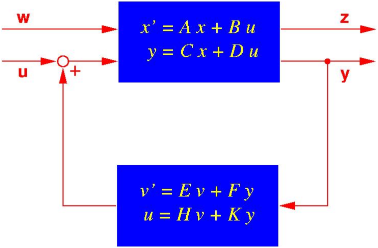
(Optimal) Control

Feedback Controllers

A feedback controller (dynamic compensator) is a linear system of order N, where

- input = output of plant,
- output = input of plant.

Modern (LQG-/$\mathcal{H}_2$-/$\mathcal{H}_\infty$-) control design: $N \geq n$.



Practical controllers require small N ($N \sim 10$, say) due to

- real-time constraints,
- increasing fragility for larger N.

$\Longrightarrow$ reduce order of plant (n) and/or controller (N).

Standard MOR techniques in systems and control: balanced truncation and related methods.

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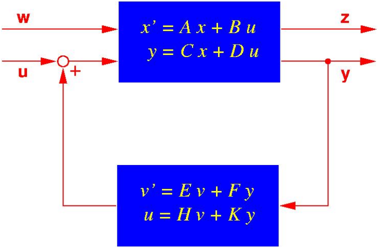

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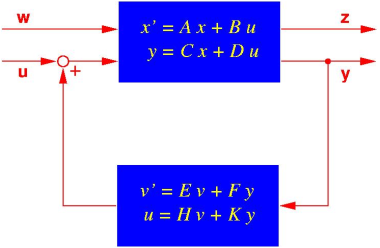

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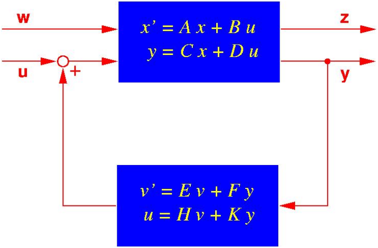

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Micro Electronics/Circuit Simulation

- Progressive miniaturization: **Moore's Law** states that the number of on-chip transistors doubles each 12 (now: 18) months.
- Verification of VLSI/ULSI chip design requires high number of simulations for different input signals.
- Increase in packing density requires modeling of interconncet to ensure that thermic/electro-magnetic effects do not disturb signal transmission.
- Linear systems in micro electronics occur through modified nodal analysis (MNA) for RLC networks, e.g., when
  - decoupling large linear subcircuits,
  - modeling transmission lines (interconnect, powergrid), parasitic effects,
  - modeling pin packages in VLSI chips,
  - modeling circuit elements described by Maxwell's equation using partial element equivalent circuits (PEEC).

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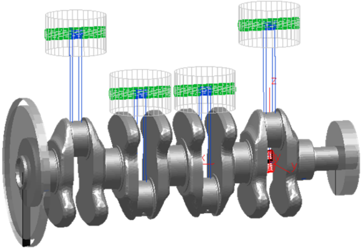

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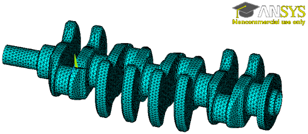

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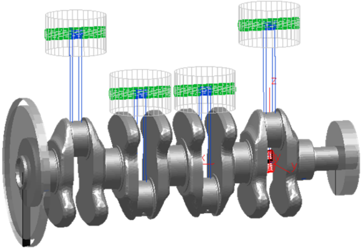

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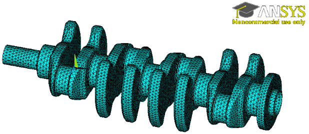

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Motivation: Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory: $4 \cdot n_x \cdot n_y$ bytes.

Theorem: (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
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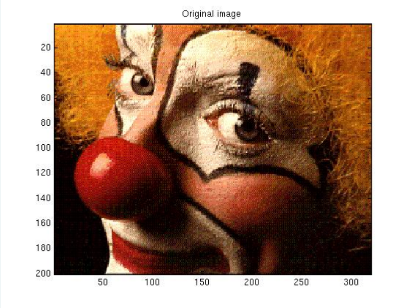
Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kb

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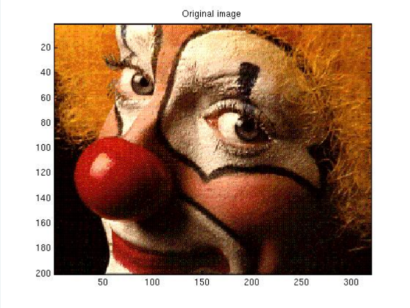

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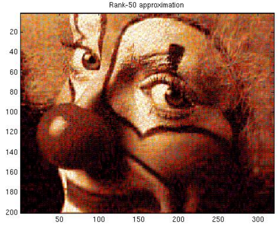

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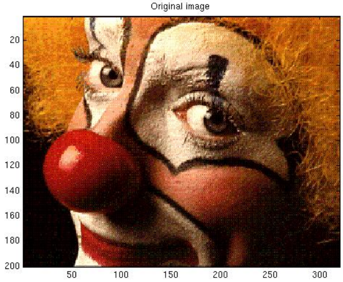

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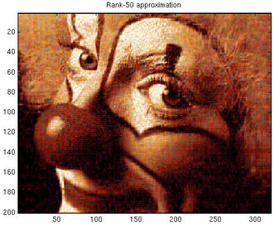
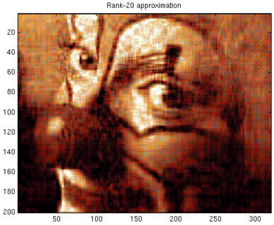


- rank $r = 20$, ≈ 42 kb



Dimension Reduction via SVD

Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,

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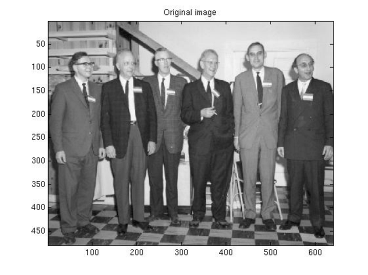

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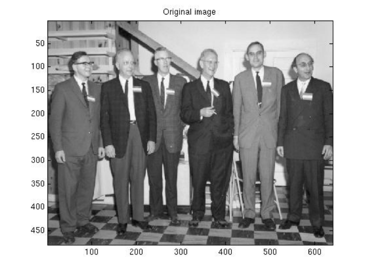
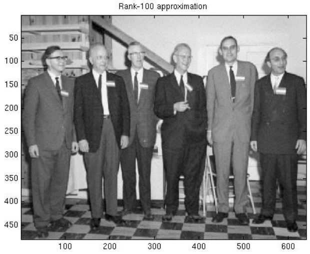
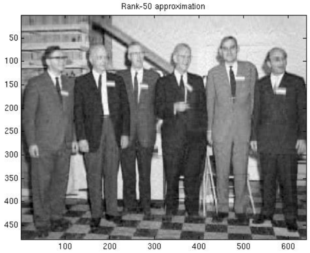


640×480 pixel, ≈ 1229 kb

- rank $r = 100$, ≈ 448 kb



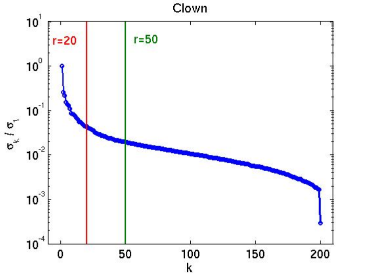
- rank $r = 50$, ≈ 224 kb

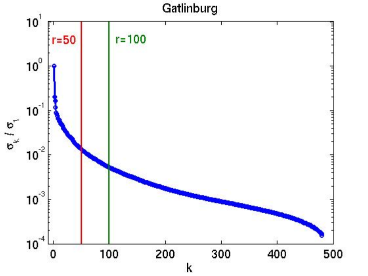


Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices





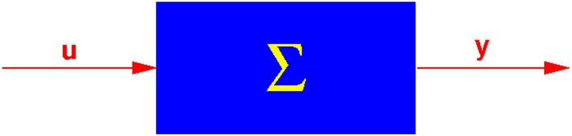
Model Reduction for Dynamical Systems

Dynamical Systems

$$\Sigma : \begin{cases} \dot{x}(t) = f(t, x(t), u(t)), & x(t_0) = x_0, \\ y(t) = g(t, x(t), u(t)) \end{cases}$$

with

- states $x(t) \in \mathbb{R}^n$,
- inputs $u(t) \in \mathbb{R}^m$,
- outputs $y(t) \in \mathbb{R}^p$.



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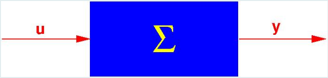

Reduced-Order System

$$\widehat{\Sigma} : \begin{cases} \dot{\hat{x}}(t) = \widehat{f}(t, \hat{x}(t), u(t)), \\ \hat{y}(t) = \widehat{g}(t, \hat{x}(t), u(t)). \end{cases}$$

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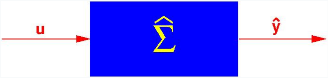

Goal:

$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

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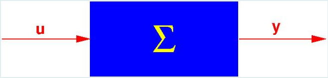

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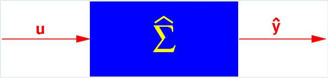

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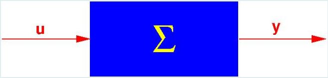

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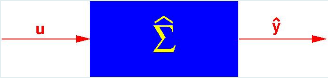

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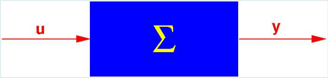

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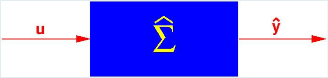

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$\|y - \hat{y}\| <$ tolerance $\cdot \|u\|$ for all admissible input signals.

Secondary goal: reconstruct approximation of x from $\hat{x}$.

Model Reduction for Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned}\dot{x} &= f(t,x,u) = Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m},\\ y &= g(t,x,u) = Cx + Du, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}.\end{aligned}$$

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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

Model Reduction for Linear Systems

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= f(t,x,u) &= Ax + Bu, \quad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= g(t,x,u) &= Cx + Du, \quad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

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- Basic Idea: use SVD approximation as for matrix A!
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Alternative to State-Space Operator: Hankel operator

Instead of

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)} Bu(\tau)\, d\tau \quad \text{for all } t \in \mathbb{R}.$$

use Hankel operator

$$\mathcal{H} : u_{-} \mapsto y_{+}, \quad y_{+}(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)} Bu(\tau)\, d\tau \quad \text{for all } t > 0.$$

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$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0.$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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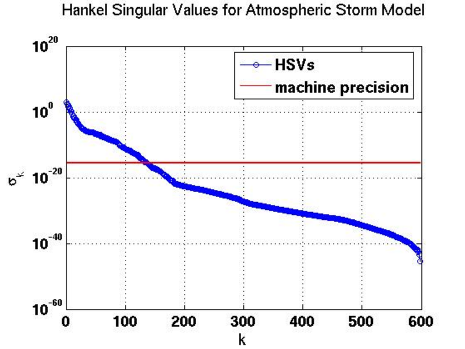

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But: computationally unfeasible for large-scale systems.

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Linear Systems in Frequency Domain

Linear, Time-Invariant (LTI) Systems

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Assumptions: $t_0 = 0$, $x_0 = x(0) = 0$.

Laplace Transform / Frequency Domain

Application of Laplace transform

$$\mathcal{L} : x(t) \mapsto x(s) = \int_0^\infty e^{-st} x(t)\, dt \quad (\Rightarrow \ \dot{x}(t) \mapsto sx(s))$$

with $s \in \mathbb{C}$ leads to linear system of equations:

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s).$$

Linear Systems in Frequency Domain

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Laplace Transform / Frequency Domain

$$sx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s)$$

yields I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sI_n - A)^{-1}B + D}_{=:G(s)}\Big)u(s) = G(s)u(s).$$

G is the transfer function of Σ, $G : \mathcal{L}_2^m \to \mathcal{L}_2^p \quad (\mathcal{L}_2 := \mathcal{L}(L_2(-\infty,\infty)))$.

Model Reduction as Approximation Problem

Approximation Problem

Approximate the dynamical system

$$\begin{array}{rcl} \dot{x} & = & Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{array}$$

by reduced-order system

$$\begin{array}{rcl} \dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, \qquad \hat{A} \in \mathbb{R}^{r\times r}, \quad \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, \qquad \hat{C} \in \mathbb{R}^{p\times r}, \quad \hat{D} \in \mathbb{R}^{p\times m}. \end{array}$$

of order $r \ll n$, such that

$$\|y - \hat{y}\| = \|Gu - \hat{G}u\| \leq \|G - \hat{G}\|\|u\| \leq \text{tolerance} \cdot \|u\|.$$

$\Longrightarrow$ Approximation problem: $\min_{\text{order}(\hat{G}) \leq r} \|G - \hat{G}\|$.

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Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u^*(\jmath\omega)u(\jmath\omega)\,d\omega.$$

Assume A is (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \mathrm{re}(z) < 0\}$.

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Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \le M \le \infty \Rightarrow$

$$\int_{-\infty}^{\infty} y^*(\jmath\omega)y(\jmath\omega)\,d\omega \quad = \quad \int_{-\infty}^{\infty} u^*(\jmath\omega)G^*(\jmath\omega)G(\jmath\omega)u(\jmath\omega)\,d\omega$$

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(Here:, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

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&= M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^* u(\jmath\omega)\, d\omega \;<\; \infty.
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$$\Longrightarrow y \in L_2^p(-\infty, \infty) \cong \mathcal{L}_2^p.$$

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Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega \in \mathbb{R}} \sigma_{max}\,(G(\jmath\omega))\,.$$

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Sketch of proof:
$\|G(\jmath\omega)u(\jmath\omega)\| \leq \|G(\jmath\omega)\|\|u(\jmath\omega)\| \Rightarrow$ "$\leq$".
Construct u with $\|Gu\|_2 = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\|\|u\|_2$.

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{re} s>0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty,\infty) \cong \mathcal{L}_2, \quad L_2(0,\infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}\hat{u}\|_2 \le \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\|F\|_2 := \left(\sup_{\operatorname{re}\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F\,d\omega\right)^{\frac{1}{2}} = \left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F\,d\omega\right)^{\frac{1}{2}}.$$

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$$\|F\|_2 \quad = \quad \left(\int_{-\infty}^{\infty} \|F(\jmath\omega)\|_F \, d\omega \right)^{\frac{1}{2}}.$$

$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \le \|G - \hat{G}\|_2 \|u_0\|.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

Qualitative and Quantitative Study of the Approximation Error

Approximation Problems

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Qualitative and Quantitative Study of the Approximation Error

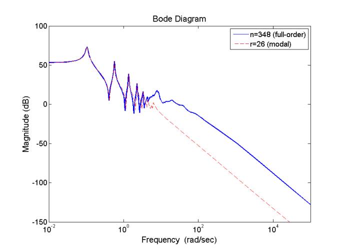
Computable error measures

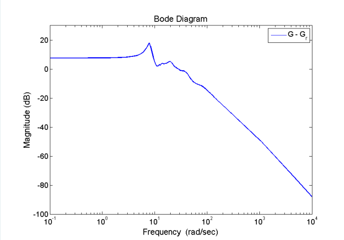
Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}$(value).

  For MIMO systems, $p \times m$ array of of plots G_{ij}.





Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

  $\Longrightarrow$ Need computable error bound/estimate!
- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

  (*"system does not generate energy"*).

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Model Reduction by Projection

Linear Algebra Basics

Projector

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r \,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

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Properties:

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector. (aka: Petrov-Galerkin projection.)
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\,w_1, \dots, w_r\,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^TV)^{-1}W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

Model Reduction by Projection

MOR Methods Based on Projection

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

Model Reduction by Projection

MOR Methods Based on Projection

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Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Model Reduction by Projection

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Important observations:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T \left(VW^T \dot{x} - AVW^T x - Bu\right)$$

Model Reduction by Projection

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Model Reduction by Projection

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Model Reduction by Projection

MOR Methods Based on Projection

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

Model Reduction by Projection

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Model Reduction by Projection
MOR Methods Based on Projection

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$P(s)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s)) \subset \mathrm{range}\,(V)$, all matrices have full rank $\Rightarrow$ "$=$", and

$$P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)$$

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Model Reduction by Projection

MOR Methods Based on Projection

Projection ⇝ Rational Interpolation

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$P(s)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A})\right)$,

if $(s_* I_n - A)^{-1}B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1}B = 0$,

hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

Model Reduction by Projection

MOR Methods Based on Projection

Projection $\rightsquigarrow$ Rational Interpolation

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$$\text{Analogously,} \quad = C(sI_n - A)^{-1}\big(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\big)B.$$

$Q(s)^*$ is a projector onto $\mathcal{W}$ $\Longrightarrow$ Given $s_* \in \mathbb{C} \setminus \left(\Lambda(A) \cup \Lambda(\hat{A}) \right)$,

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$

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Model Reduction by Projection

MOR Methods Based on Projection

Theorem [Grimme '97, Villemagne/Skelton '87]

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \mathrm{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \mathrm{range}(W)$,

then the interpolation condition

$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Note: extension to Hermite interpolation conditions later!

Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A-invariant subspace $\mathcal{V} = \mathrm{span}(t_1, \ldots, t_r)$, v_k = eigenvectors corresp. to "dominant" modes / eigenvalues of A. Then with

$$V = T(:, 1:r) = [\, t_1, \ldots, t_r \,], \quad \tilde{W} = T^{-1}(:, 1:r), \quad W = \tilde{W}(V^* \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^* A V = \mathrm{diag}\, \{\lambda_1, \ldots, \lambda_r\}, \quad \hat{B} := W^* B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\, \hat{C}, \ C_2 \,], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \leq \|C_2\|\|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\,\hat{C},\ C_2\,]\begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix}\begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

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$$\|G - \hat{G}\|_\infty \le \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\mathrm{Re}(\lambda)|}.$$

Proof:

$$G(s) \quad = \quad \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega\in\mathbb{R}} \sigma_{\max}(C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(\jmath\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \mathrm{diag}\left(\frac{1}{\jmath\omega - \lambda_{r+1}}, \ldots, \frac{1}{\jmath\omega - \lambda_n}\right) B_2.$$

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Difficulties:

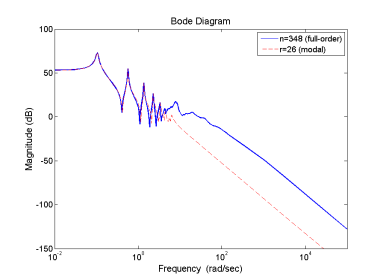
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
  ([Litz '79] use Jordan canoncial form; otherwise merely heuristic criteria, e.g., [Varga '95]. Recent improvement: dominant pole algorithm.)
- Error bound not computable for really large-scale problems.

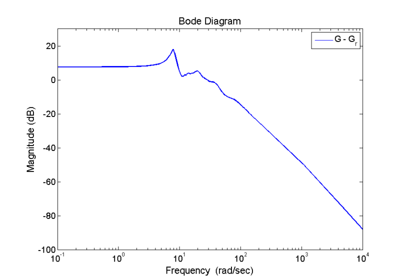
Modal Truncation

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = p = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function





MATLAB® demo.

Coffee break!

Modal Truncation

Example

BEAM, SISO system from SLICOT Benchmark Collection for Model Reduction, $n = 348$, $m = p = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_\infty \leq 1.21 \cdot 10^3$

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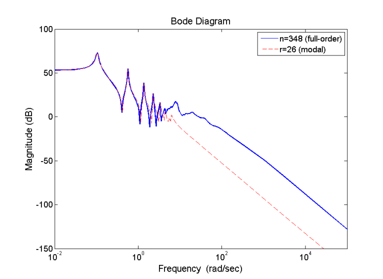

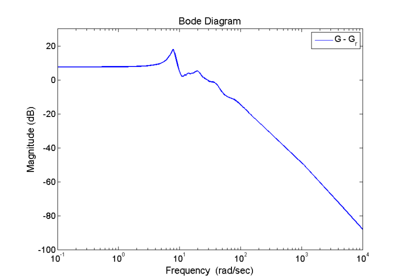

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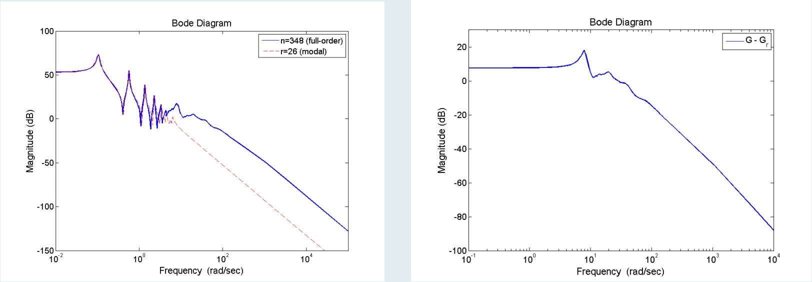

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Balanced Truncation

Basic principle:

- A system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$.

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$$\mathcal{H}^*\mathcal{H}u(t) \;=\; B^T e^{-A^T t} \int_0^{\infty} e^{A^T \tau} C^T Ce^{A\tau} z\, d\tau$$

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$\Longleftrightarrow \quad PQz = \sigma^2 z. \quad \square$

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- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) \quad &\mapsto \quad (TAT^{-1}, TB, CT^{-1}, D) \\ &= \quad \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

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Balanced Truncation

Motivation:

HSVs are system invariants: they are preserved under
$\mathcal{T} : (A, B, C, D) \mapsto (TAT^{-1}, TB, CT^{-1}, D)$:

in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow \quad (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda\,(PQ) = \Lambda\,((TPT^T)(T^{-T}QT^{-1}))$.

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HSVs are system invariants: they are preserved under
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HSVs determine the energy transfer given by the Hankel map

$$\mathcal{H} : L_2(-\infty, 0) \mapsto L_2(0, \infty) : u_- \mapsto y_+.$$

In balanced coordinates . . . energy transfer from u_- to y_+:

$$E := \sup_{\substack{u \in L_2(-\infty,0] \\ x(0)=x_0}} \frac{\int_0^\infty y(t)^T y(t)\,dt}{\int_{-\infty}^0 u(t)^T u(t)\,dt} = \frac{1}{\|x_0\|_2} \sum_{j=1}^n \sigma_j^2 x_{0,j}^2$$

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$\Longrightarrow$ Truncate states corresponding to "small" HSVs
$\Longrightarrow$ complete analogy to best approximation via SVD!

Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

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Note:

$$V^T W \;=\; (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}})$$

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$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

Note:

$$\begin{aligned} V^T W &= (\Sigma_1^{-\frac{1}{2}} U_1^T S)(R^T V_1 \Sigma_1^{-\frac{1}{2}}) = \Sigma_1^{-\frac{1}{2}} U_1^T U \Sigma V^T V_1 \Sigma_1^{-\frac{1}{2}} \\ &= \Sigma_1^{-\frac{1}{2}} [\,I_r,\, 0\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} I_r \\ 0 \end{bmatrix} \Sigma_1^{-\frac{1}{2}} \end{aligned}$$

Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.
2. Compute SVD $SR^T = [\, U_1,\ U_2\,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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Balanced Truncation

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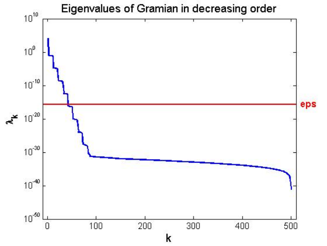
"New" algorithmic ideas from numerical linear algebra:

– Instead of Gramians P, Q compute $S, R \in \mathbb{R}^{n \times k}$, $k \ll n$, such that

$$P \approx SS^T, \quad Q \approx RR^T.$$

– Compute S, R with problem-specific Lyapunov solvers of "low" complexity directly.



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"New" algorithmic ideas from numerical linear algebra:

Sparse Balanced Truncation:

- Sparse implementation using sparse Lyapunov solver (→ADI+MUMPS/SuperLU).
- Complexity $\mathcal{O}(n(k^2 + r^2))$.
- Software:
  - + MATLAB toolbox LyaPack (Penzl 1999),
  - + Software library M.E.S.S.[a] in C/MATLAB [B./Saak/Köhler].

[a] Matrix Equation Sparse Solvers

ADI Methods for Lyapunov Equations
Background

Recall Peaceman Rachford ADI:
Consider $Au = s$ where $A \in \mathbb{R}^{n \times n}$ spd, $s \in \mathbb{R}^n$. ADI Iteration Idea:
Decompose $A = H + V$ with $H, V \in \mathbb{R}^{n \times n}$ such that

$$(H + pI)v = r$$
$$(V + pI)w = t$$

can be solved easily/efficiently.

ADI Iteration

If H, V spd $\Rightarrow \exists p_k,\ k = 1, 2, \ldots$ such that

$$\begin{aligned} u_0 &= 0 \\ (H + p_k I)u_{k-\frac{1}{2}} &= (p_k I - V)u_{k-1} + s \\ (V + p_k I)u_k &= (p_k I - H)u_{k-\frac{1}{2}} + s \end{aligned}$$

converges to $u \in \mathbb{R}^n$ solving $Au = s$.

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ADI Methods for Lyapunov Equations

The Lyapunov operator

$$\mathcal{L}: \quad P \quad \mapsto \quad AX + XA^T$$

can be decomposed into the linear operators

$$\mathcal{L}_H : X \mapsto AX, \qquad \mathcal{L}_V : X \mapsto XA^T.$$

In analogy to the standard ADI method we find the

ADI iteration for the Lyapunov equation [Wachspress '88]

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -W - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -W - X_{k-\frac{1}{2}}^T(A^T - p_k I). \end{aligned}$$

ADI Methods for Lyapunov Equations

Low-Rank ADI

Consider $AX + XA^T = -BB^T$ for stable A, $B \in \mathbb{R}^{n\times m}$ with $m \ll n$.

ADI iteration for the Lyapunov equation [Wachspress '95]

For $k = 1, \ldots, k_{\max}$

$$\begin{aligned} X_0 &= 0 \\ (A + p_k I)X_{k-\frac{1}{2}} &= -BB^T - X_{k-1}(A^T - p_k I) \\ (A + p_k I)X_k^T &= -BB^T - X_{k-\frac{1}{2}}^T(A^T - p_k I) \end{aligned}$$

Rewrite as one step iteration and factorize $X_k = Z_k Z_k^T$, $k = 0, \ldots, k_{\max}$

$$\begin{aligned} Z_0 Z_0^T &= 0 \\ Z_k Z_k^T &= -2p_k(A + p_k I)^{-1}BB^T(A + p_k I)^{-T} \\ &\quad +(A + p_k I)^{-1}(A - p_k I)Z_{k-1}Z_{k-1}^T(A - p_k I)^T(A + p_k I)^{-T} \end{aligned}$$

$\ldots \rightsquigarrow$ low-rank Cholesky factor ADI

[Penzl '97/'00, Li/White '99/'02, B./Li/Penzl '99/'08, Gugercin/Sorensen/Antoulas '03]

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Balanced Truncation

ADI Methods for Lyapunov Equations

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B, \ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$

[PENZL '00]

Observing that $(A - p_i I)$, $(A + p_k I)^{-1}$ commute, we rewrite $Z_{k_{\max}}$ as

$$Z_{k_{\max}} = [z_{k_{\max}}, \ P_{k_{\max}-1} z_{k_{\max}}, \ P_{k_{\max}-2}(P_{k_{\max}-1} z_{k_{\max}}), \ \ldots, \ P_1(P_2 \cdots P_{k_{\max}-1} z_{k_{\max}})],$$

[LI/WHITE '02]

where

$$z_{k_{\max}} = \sqrt{-2p_{k_{\max}}}(A + p_{k_{\max}} I)^{-1} B$$

and

$$P_i := \frac{\sqrt{-2p_i}}{\sqrt{-2p_{i+1}}} \left[I - (p_i + p_{i+1})(A + p_i I)^{-1} \right].$$

Balanced Truncation

ADI Methods for Lyapunov Equations

$$Z_k = [\sqrt{-2p_k}(A + p_k I)^{-1}B, \ (A + p_k I)^{-1}(A - p_k I)Z_{k-1}]$$

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ADI Methods for Lyapunov Equations

Lyapunov equation $0 = AX + XA^T + BB^T$.

Algorithm [Penzl '97/'00, Li/White '99/'02, B. 04, B./Li/Penzl '99/'08]

$$V_1 \quad \leftarrow \quad \sqrt{-2\,\mathrm{re}\,p_1}(A + p_1 I)^{-1} B, \qquad Z_1 \leftarrow V_1$$

FOR $k = 2, 3, \ldots$

$$V_k \leftarrow \sqrt{\frac{\mathrm{re}\,p_k}{\mathrm{re}\,p_{k-1}}}\left(V_{k-1} - (p_k + \overline{p_{k-1}})(A + p_k I)^{-1} V_{k-1}\right)$$

$$Z_k \leftarrow \left[\begin{array}{cc} Z_{k-1} & V_k \end{array}\right]$$

$$Z_k \leftarrow \mathrm{rrlq}(Z_k, \tau) \qquad \text{column compression}$$

At convergence, $Z_{k_{\max}} Z_{k_{\max}}^T \approx X$, where (without column compression)

$$Z_{k_{\max}} = \left[\begin{array}{ccc} V_1 & \ldots & V_{k_{\max}} \end{array}\right], \quad V_k = \left[\;\right] \in \mathbb{C}^{n \times m}.$$

Note: Implementation in real arithmetic possible by combining two steps [B./Li/Penzl '99/'08] or using new idea employing the relation of 2 consecutive complex factors [B./Kürschner/Saak '11].

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Numerical Results for ADI

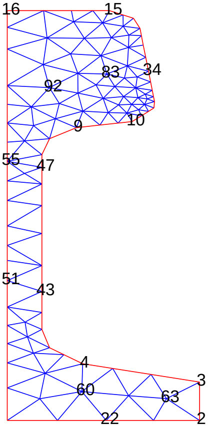
Optimal Cooling of Steel Profiles

- Mathematical model: boundary control for linearized 2D heat equation.

$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, & \xi \in \Omega \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), & \xi \in \Gamma_k,\ 1 \le k \le 7, \\
\frac{\partial}{\partial n} x &= 0, & \xi \in \Gamma_7.
\end{aligned}
$$

$\Longrightarrow m = 7, p = 6.$

- FEM Discretization, different models for initial mesh ($n = 371$), 1, 2, 3, 4 steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209, 79841.$



Source: Physical model: courtesy of Mannesmann/Demag.

Math. model: Tröltzsch/Unger 1999/2001, Penzl 1999, Saak 2003.

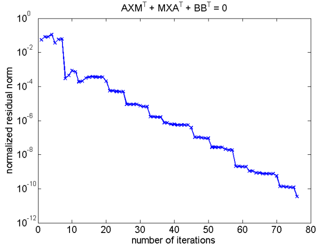
Numerical Results for ADI

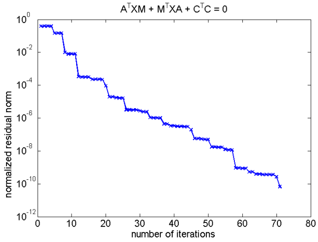
Optimal Cooling of Steel Profiles

- Solve dual Lyapunov equations needed for balanced truncation, i.e.,

$$APM^T + MPA^T + BB^T = 0, \quad A^TQM + M^TQA + C^TC = 0,$$

for 79,841.
- 25 shifts chosen by Penzl heuristic from 50/25 Ritz values of A of largest/smallest magnitude, no column compression performed.
- New version in M.E.S.S. requires no factorization of mass matrix!
- Computations done on Core2Duo at 2.8GHz with 3GB RAM and 32Bit-MATLAB.





CPU times: 626 / 356 sec.

Numerical Results for ADI

Scaling / Mesh Independence **Computations by Martin Köhler '10**

- $A \in \mathbb{R}^{n \times n} \equiv$ FDM matrix for 2D heat equation on $[0, 1]^2$ (LYAPACK benchmark `demo_l1`, $m = 1$).
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CPU Times

n	M.E.S.S. (C)	LyaPack	M.E.S.S. (MATLAB)
100	0.023	0.124	0.158
625	0.042	0.104	0.227
2,500	0.159	0.702	0.989
10,000	0.965	6.22	5.644
40,000	11.09	71.48	34.55
90,000	34.67	418.5	90.49
160,000	109.3	out of memory	219.9
250,000	193.7	out of memory	403.8
562,500	930.1	out of memory	1216.7
1,000,000	2220.0	out of memory	2428.6

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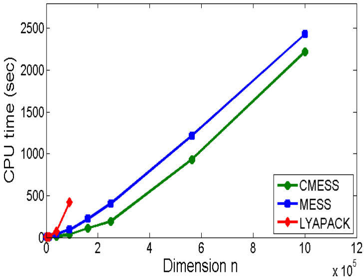

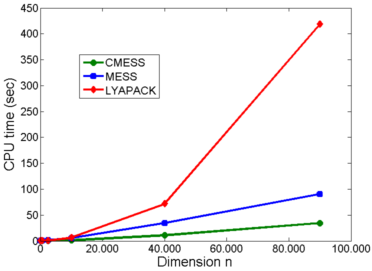

Note: for $n = 1,000,000$, first sparse LU needs $\sim 1,100$ sec., using UMFPACK this reduces to 30 sec.

Factored Galerkin-ADI Iteration

Lyapunov equation $0 = AX + XA^T + BB^T$

Projection-based methods for Lyapunov equations with $A + A^T < 0$:

1. Compute orthonormal basis $\mathrm{range}(Z)$, $Z \in \mathbb{R}^{n\times r}$, for subspace $\mathcal{Z} \subset \mathbb{R}^n$, $\dim \mathcal{Z} = r$.
2. Set $\hat{A} := Z^T AZ$, $\hat{B} := Z^T B$.
3. Solve small-size Lyapunov equation $\hat{A}\hat{X} + \hat{X}\hat{A}^T + \hat{B}\hat{B}^T = 0$.
4. Use $X \approx Z\hat{X}Z^T$.

Examples:

- Krylov subspace methods, i.e., for $m = 1$:

$$\mathcal{Z} = \mathcal{K}(A, B, r) = \mathrm{span}\{B, AB, A^2B, \ldots, A^{r-1}B\}$$

[Saad '90, Jaimoukha/Kasenally '94, Jbilou '02–'08].

- K-PIK [Simoncini '07],

$$\mathcal{Z} = \mathcal{K}(A, B, r) \cup \mathcal{K}(A^{-1}, B, r).$$

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Examples:

- ADI subspace [B./R.-C. Li/Truhar '08]:

$$\mathcal{Z} = \mathrm{colspan}\left[\begin{array}{ccc} V_1, & \ldots, & V_r \end{array}\right].$$

Note:

1. ADI subspace is rational Krylov subspace [J.-R. Li/White '02].
2. Similar approach: ADI-preconditioned global Arnoldi method [Jbilou '08].

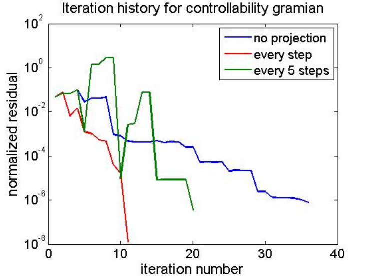
Factored Galerkin-ADI Iteration

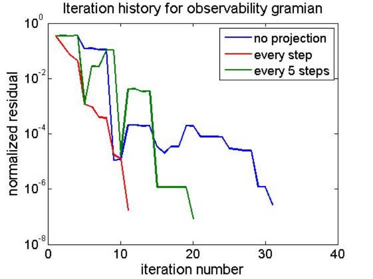
Numerical examples for Galerkin-ADI

FEM semi-discretized control problem for parabolic PDE:

- optimal cooling of rail profiles,
- $n = 20,209,\ m = 7,\ p = 6.$

Good ADI shifts





CPU times: 80s (projection every 5th ADI step) vs. 94s (no projection).

Computations by Jens Saak '10.

Factored Galerkin-ADI Iteration

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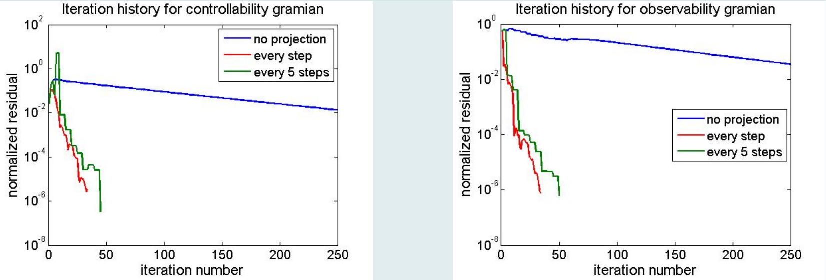


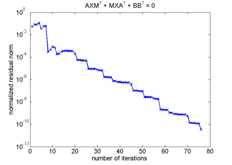
CPU times: 368s (projection every 5th ADI step) vs. 1207s (no projection).

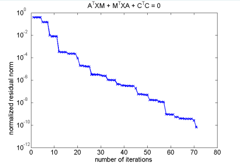
Computations by Jens Saak '10.

Factored Galerkin-ADI Iteration

Numerical examples for Galerkin-ADI: optimal cooling of rail profiles, $n = 79,841$.

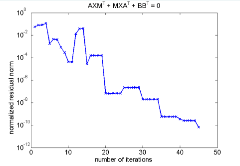
MESS w/o Galerkin projection and column compression

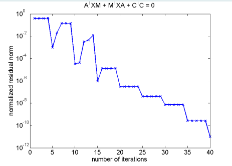




Rank of solution factors: 532 / 426

MESS with Galerkin projection and column compression



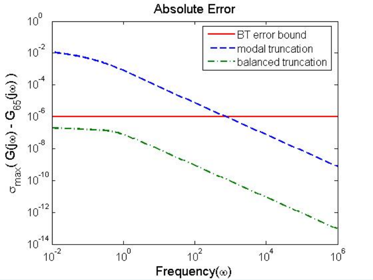


Rank of solution factors: 269 / 205

Balanced Truncation

Numerical example for BT: Optimal Cooling of Steel Profiles

$n = 1357$, Absolute Error



- BT model computed with sign function method,
- MT w/o static condensation, same order as BT model.

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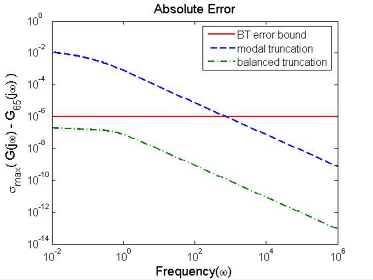
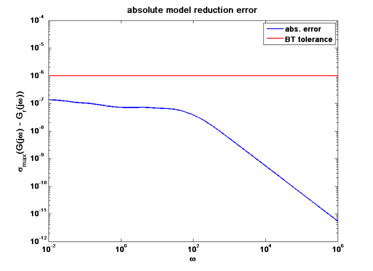


– BT model computed with sign function method,

– MT w/o static condensation, same order as BT model.

$n = 79841$, Absolute Error



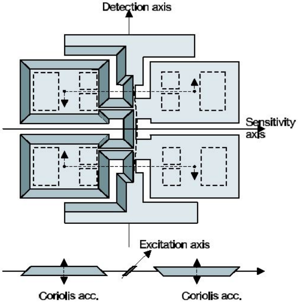
– BT model computed using M.E.S.S. in MATLAB,

– dualcore, computation time: <10 min.

Balanced Truncation

Numerical example for BT: Microgyroscope (Butterfly Gyro)

- Vibrating micro-mechanical gyroscope for inertial navigation.
- Rotational position sensor.



- By applying AC voltage to electrodes, wings are forced to vibrate in anti-phase in wafer plane.
- Coriolis forces induce motion of wings out of wafer plane yielding sensor data.

Source: The Oberwolfach Benchmark Collection `http://www.imtek.de/simulation/benchmark`

Courtesy of D. Billger (Imego Institute, Göteborg), Saab Bofors Dynamics AB.

Balanced Truncation

Numerical example for BT: Microgyroscope (Butterfly Gyro)

- FEM discretization of structure dynamical model using quadratic tetrahedral elements (ANSYS-SOLID187)
$\rightsquigarrow$ $n = 34,722$, $m = 1$, $p = 12$.
- Reduced model computed using SpaRed, $r = 30$.

Balanced Truncation

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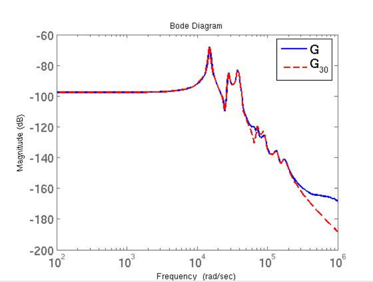
Frequency Repsonse Analysis



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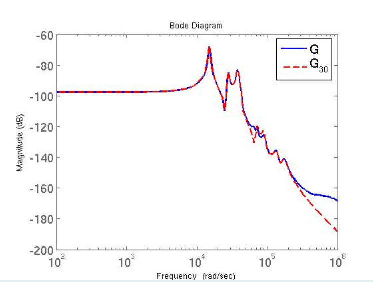
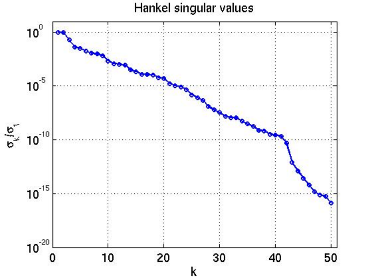


Hankel Singular Values



Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [Mullis/Roberts '76, Moore '81]

- P = controllability Gramian of system given by (A, B, C, D).
- Q = observability Gramian of system given by (A, B, C, D).
- P, Q solve dual Lyapunov equations

$$AP + PA^T + BB^T \ = \ 0, \qquad A^T Q + QA + C^T C \ = \ 0.$$

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Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

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and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

LQG Balanced Truncation (LQGBT) [Jonckheere/Silverman '83]

- $P/Q =$ controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual algebraic Riccati equations (AREs)

$$\begin{aligned} 0 &= AP + PA^T - PC^T CP + B^T B, \\ 0 &= A^T Q + QA - QBB^T Q + C^T C. \end{aligned}$$

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST) [Desai/Pal '84, Green '88]

- $P =$ controllability Gramian of system given by (A, B, C, D), i.e., solution of Lyapunov equation $AP + PA^T + BB^T = 0$.
- $Q =$ observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D), i.e., solution of ARE

$$\hat{A}^T Q + Q\hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT) [Green '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual AREs

$$\begin{aligned} 0 &= \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T, \\ 0 &= \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C, \end{aligned}$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Model Reduction

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \mathrm{diag}(\sigma_1, \ldots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \ldots \geq \sigma_n \geq 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [Opdenacker/Jonckheere '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [Mustafa/Glover '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Model Reduction

Properties

- Guaranteed preservation of physical properties like
  - stability (all),
  - passivity (PRBT),
  - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT:} \quad \|G - G_r\|_\infty \le 2 \sum_{j=r+1}^{n} \sigma_j^{BT},$$

$$\text{LQGBT:} \quad \|G - G_r\|_\infty \le 2 \sum_{j=r+1}^{n} \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST:} \quad \|G - G_r\|_\infty \le \left(\prod_{j=r+1}^{n} \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

- Can be combined with singular perturbation approximation for steady-state performance.
- Computations can be modularized.

Padé Approximation

Idea:

- Consider
$$\dot{x} = Ax + Bu, \quad y = Cx$$
with transfer function $G(s) = C(sI_n - A)^{-1}B$.
- For $s_0 \notin \Lambda(A)$:

$$\begin{aligned} G(s) &= C\left(I - (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\ &= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \dots \end{aligned}$$

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- For $s_0 \notin \Lambda(A)$:
$$\begin{aligned} G(s) &= C((s_0 I_n - A) + (s - s_0)I_n)^{-1}B \\ &= C\left(I - (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\ &= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots \end{aligned}$$

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  - For $s_0 = 0$: $m_j := C(A^{-1})^j B =$ moments.
  - For $s_0 = \infty$: $m_j := CA^{j-1}B =$ Markov parameters.

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- As reduced-order model use rth Padé approximant $\hat{G}$ to G:

$$G(s) = \hat{G}(s) + \mathcal{O}((s-s_0)^{2r}),$$

i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r-1$

$\rightsquigarrow$ moment matching if $s_0 < \infty$,

$\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}\tilde{B}) =: \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0 I_n - A)^{-1}$, $\tilde{B} = (s_0 I_n - A)^{-1} B$) along

$$\mathcal{W} = \mathrm{span}(C^T, \tilde{A}^* C^T, \ldots, (\tilde{A}^*)^{r-1} C^T) =: \mathcal{K}(\tilde{A}^*, C^T, r).$$

- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.

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- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.

Remark: Arnoldi (PRIMA) yields only $G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^r)$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

Difficulties:

- Computable error estimates/bounds for $\|y - \hat{y}\|_2$ often very pessimistic or expensive to evaluate.
- Mostly heuristic criteria for choice of expansion points.
  Optimal choice for second-order systems with proportional/Rayleigh damping (Beattie/Gugercin '05).
- Good approximation quality only locally.
- Preservation of physical properties only in special cases; usually requires post processing which (partially) destroys moment matching properties.

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Interpolatory Model Reduction

Short Introduction

Computation of reduced-order model by projection

Given an LTI system $\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B,$ a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$ by computing

$$\hat{A} = W^T A V, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

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$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Interpolatory Model Reduction

Short Introduction

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned}\mathrm{span}\left\{(s_1 I_n - A)^{-1}B, \ldots, (s_k I_n - A)^{-1}B\right\} &\subset \mathrm{Ran}(V),\\ \mathrm{span}\left\{(s_1 I_n - A)^{-T}C^T, \ldots, (s_k I_n - A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W),\end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

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then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

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then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

Remarks:

$k = 1$, standard Krylov subspace(s) of dimension $K \rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1) = \frac{d^i}{ds^i}\hat{G}(s_1), \quad i = 0, \ldots, K-1(+K).$$

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iterative Rational Krylov-Algo. [Antoulas/Beattie/Gugercin '07].

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

$$\text{Find} \quad \arg\min_{\hat{G} \in \mathcal{H}_2 \text{ of order } \leq r} \| G - \hat{G} \|_2 .$$

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

Find $\arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$

$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For SISO systems

$$G(-\mu_i) = \hat{G}(-\mu_i),$$
$$G'(-\mu_i) = \hat{G}'(-\mu_i),$$

where μ_i are the poles of the reduced transfer function $\hat{G}$.

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

Find $\arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$

$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For MIMO systems

$$\begin{aligned} G(-\mu_i)\tilde{B}_i &= \hat{G}(-\mu_i)\tilde{B}_i, && \text{for } i = 1,\ldots,r,\\ \tilde{C}_i^T G(-\mu_i) &= \tilde{C}_i^T \hat{G}(-\mu_i), && \text{for } i = 1,\ldots,r,\\ \tilde{C}_i^T G'(-\mu_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\mu_i)\tilde{B}_i, && \text{for } i = 1,\ldots,r, \end{aligned}$$

where $T^{-1}\hat{A}T = \operatorname{diag}\{\mu_1,\ldots,\mu_r\}$ = spectral decomposition and

$$\tilde{B} = \hat{B}^T T^{-T}, \quad \tilde{C} = \hat{C}T.$$

$\rightsquigarrow$ tangential interpolation conditions.

Interpolatory Model Reduction

Interpolation of the Transfer Function by Projection

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^TAV\right)^{-1}W^TB,$$

where V and W are given as the rational Krylov subspaces

$$V = \left[(-\mu_1 I - A)^{-1}B, \ldots, (-\mu_r I - A)^{-1}B\right],$$
$$W = \left[(-\mu_1 I - A^T)^{-1}C^T, \ldots, (-\mu_r I - A^T)^{-1}C^T\right].$$

Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$ as desired.

$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. '06], [Bunse-Gerstner et al. '07], [Van Dooren et al. '08]

Interpolatory Model Reduction

Interpolation of the Transfer Function by Projection

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV \left(sI - W^T AV \right)^{-1} W^T B,$$

where V and W are given as the rational Krylov subspaces

$$V = \left[(-\mu_1 I - A)^{-1} B, \ldots, (-\mu_r I - A)^{-1} B \right],$$
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$\mathcal{H}_2$-Optimal Model Reduction

The basic IRKA Algorithm

Algorithm 1 IRKA

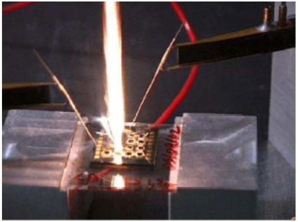
Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.
Output: A^{opt}, B^{opt}, C^{opt}

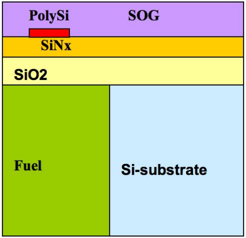
1: **while** $(\max_{j=1,\ldots,r} \left\{ \frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|} \right\} > \delta)$ **do**
2: $\quad \operatorname{diag}\{\mu_1, \ldots, \mu_r\} := T^{-1}\hat{A}T =$ spectral decomposition, $\tilde{B} = \hat{B}^* T^{-*}$, $\tilde{C} = \hat{C}T$.
3: $\quad V = \left[(-\mu_1 I - A)^{-1} B\tilde{B}_1, \ldots, (-\mu_r I - A)^{-1} B\tilde{B}_r\right]$
4: $\quad W = \left[(-\mu_1 I - A^T)^{-1} C^T\tilde{C}_1, \ldots, (-\mu_r I - A^T)^{-1} C^T\tilde{C}_r\right]$
5: $\quad V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$
6: $\quad \hat{A} = (W^*V)^{-1} W^*AV$, $\hat{B} = (W^*V)^{-1} W^*B$, $\hat{C} = CV$
7: **end while**
8: $A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$

Numerical Comparison of MOR Approaches

Microthruster

- Co-integration of solid fuel with silicon micromachined system.
- Goal: Ignition of solid fuel cells by electric impulse.
- Application: nano satellites.
- Thermo-dynamical model, ignition via heating an electric resistance by applying voltage source.
- Design problem: reach ignition temperature of fuel cell w/o firing neighbouring cells.
- Spatial FEM discretization of thermo-dynamical model $\rightsquigarrow$ linear system, $m = 1$, $p = 7$.





Source: The Oberwolfach Benchmark Collection http://www.imtek.de/simulation/benchmark

Courtesy of C. Rossi, LAAS-CNRS/EU project "Micropyros".

Numerical Comparison of MOR Approaches

Microthruster

- axial-symmetric 2D model
- FEM discretisation using linear (quadratic) elements $\rightsquigarrow n = 4,257$ $(11,445)$ $m = 1$, $p = 7$.
- Reduced model computed using SpaRed. modal truncation using ARPACK, and Z. Bai's PVL implementation.

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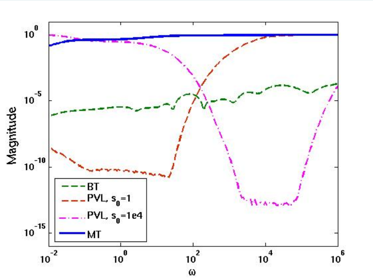

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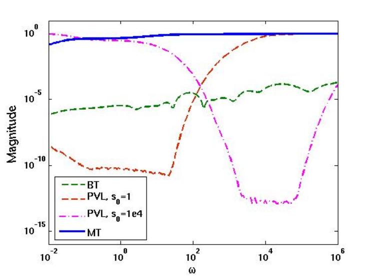

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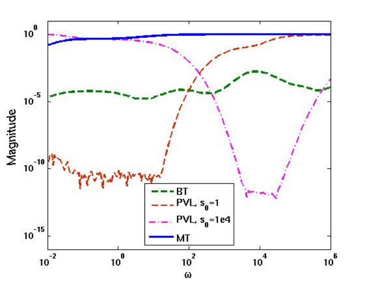

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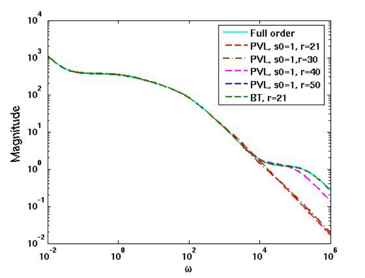
Frequency Response BT/PVL



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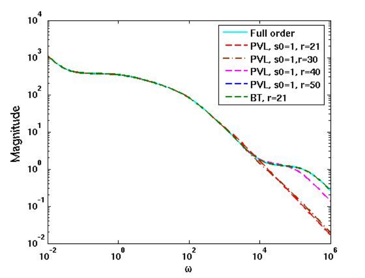
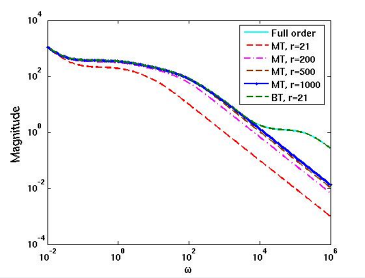


Frequency Response BT/MT



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- Order of reduced model: $r = 28$.

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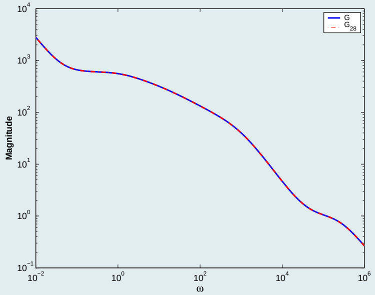
Frequency Repsonse Analysis



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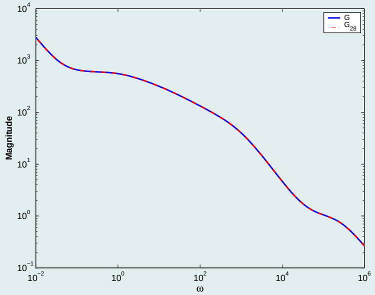
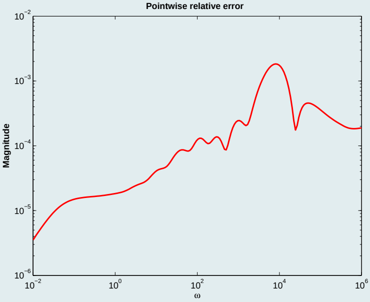


Relative Error



Topics Not Covered

- Balanced residualization (singular perturbation approximation), yields $G(0) = \hat{G}(0)$.
- Special methods for second-order (mechanical) systems.
- Extensions to bilinear and stochastic systems.
- Rational interpolation methods for nonlinear systems.
- Other MOR techniques like POD, RB.
- MOR methods for discrete-time systems.
- Extensions to descriptor systems $E\dot{x} = Ax + Bu$, E singular.
- Parametric model reduction:

$$\dot{x} = A(p)x + B(p)u, \quad y = C(p)x,$$

  where $p \in \mathbb{R}^d$ is a free parameter vector; parameters should be preserved in the reduced-order model.

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