Using model order reduction for computing fast frequency sweeps of vibro-acoustic systems described by indirect boundary element models

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Outline

1 Problem description

2 System assembly: an efficient interpolation approach
   - Frequency scaling of the system matrices
   - Frequency interpolation of the system matrices
   - Determining the frequency windows

3 System solving: computing Padé approximants (MOR)

4 Proposed algorithm

5 Numerical examples
   - Exterior application
   - Interior/exterior application

6 Conclusion
Using MOR to compute **fast frequency sweeps**...

Why frequency, why frequency sweeps and why fast sweeps?

Analyzing systems in the **frequency domain** allows one to infer properties regarding resonances (e.g., vibro-acoustic systems), filtering properties (e.g., electrical systems), etc.
Using MOR to compute fast frequency sweeps...

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Analyzing systems in the frequency domain allows one to infer properties regarding resonances (e.g., vibro-acoustic systems), filtering properties (e.g., electrical systems), etc.

A frequency sweep amounts to solving a linear system $A(f)x(f) = b(f)$ (e.g., $A(f) = j2\pi f I - A$) at many frequencies.
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A fast frequency sweep avoids solving the large linear system for each frequency by using extrapolation.
Using... of vibro-acoustic systems...

What are vibro-acoustic systems?

Vibro-acoustics or structural acoustics is the study of the acoustic waves in structures and how they interact with and radiate into adjacent media.
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The steady-state acoustic pressure generated by harmonic excitations at wavenumber $k$ is described by the scalar wave equation in 3D (Helmholtz):

$$\nabla^2 p + k^2 p = 0 \text{ in the domain } \Omega,$$
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The steady-state acoustic pressure generated by harmonic excitations at wavenumber \(k\) is described by the scalar wave equation in 3D (Helmholtz):

\[
\nabla^2 p + k^2 p = 0 \quad \text{in the domain } \Omega, \text{ where}
\]

- \(\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2}\)
- \(p\) denotes the complex amplitude of the pressure
- domain \(\Omega\) contains an inviscid compressible fluid.
What is IBEM?

Indirect formulation is employed for interior and exterior problems.

In IBEM, the unknowns are $\sigma = \frac{\partial p^+}{\partial n^+} - \frac{\partial p^-}{\partial n^-}$ (single layer potential) and $\mu = p^+ - p^-$ (double layer potential). Acoustic pressure at field point $X$ is

$$p(X) = \int_S \left( G(X, Y)\sigma(Y) - \frac{\partial G(X, Y)}{\partial n(Y)} \mu(Y) \right) dS.$$ 

$$G(X, Y) = \frac{\exp(-ikR)}{4\pi R}, \quad R = |X - Y|$$ is the 3D Green’s function.
Problem description

What is IBEM?

The surface $S$ is discretized into boundary elements $S \cong \sum S^e$.

The unknowns are expressed at the discretization points (nodes) as

$$\mu(X) = N_\mu \cdot \mu, \quad \sigma(X) = N_\sigma \cdot \sigma$$

with $\mu$ and $\sigma$, vectors of nodal double and single layer potentials, and $N_\mu$ and $N_\sigma$, shape functions. This yields the system of equations of size $N_{DOF}$:

$$
\begin{bmatrix}
A_{\sigma\sigma} & A_{\sigma\mu} \\
A_{H}^{\sigma} & A_{\mu\mu}
\end{bmatrix}
\begin{bmatrix}
\sigma \\
\mu
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
b_\mu
\end{bmatrix}.
$$

$$A_{\sigma\sigma} = -\frac{1}{ik} \sum_e \int_{S_e} \frac{N_\sigma^T(X)N_\sigma(X)}{\Delta\beta(X)} dS_e + \sum_e \int_{S_e} \sum_e \int_{S_e} N_\sigma^T(X)N_\sigma(Y)G(X,Y)dS_e dS_e.$$

$A$ is complex, symmetric, dense, has complicated frequency dependency.
Challenges and goals

Challenges for IBEM:

- assembling and solving are equally expensive.
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- assembling and solving are equally expensive.

Goals for FFS:

- avoid assembling the system matrices at each frequency: perform polynomial interpolation on appropriate frequency scaled matrices
- avoid solving a linear system at each frequency: employ Padé approximations.
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Frequency scaling of the system matrices

Scale the entries \( \hat{A}_{[m,n]} = \left\{ \begin{array}{ll} e^{ikR_{[m,n]}}A_{[m,n]} & , \quad m, n = 1, \ldots, N_{DOF} \end{array} \right. \)

Figure: The effect of applying the scaling factor on two matrix entries
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The scaled matrices are interpolated by Lagrange polynomials:

\[
\tilde{A}_{[m,n]}(k) = \sum_{j=1}^{N+1} P_j(k) \hat{A}_{[m,n]}(k_j), \quad P_j(k) = 1, \quad k = k_j, \quad \& \quad P_j(k) = 0, \quad k \neq k_j.
\]
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We call the Lagrange nodes \( k_j = \text{the master wavenumbers (frequencies)} \).
Frequency interpolation of the system matrices

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We call the Lagrange nodes \( k_j = \) the master wavenumbers (frequencies). The interpolation order \( N \) can be 1, 2, or higher. \( N + 1 \) system matrices need to be assembled and stored, so one needs to find a trade-off.
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Frequency interpolation of the system matrices

The scaled matrices are interpolated by Lagrange polynomials:

$$\tilde{A}_{[m,n]}(k) = \sum_{j=1}^{N+1} P_j(k) \tilde{A}_{[m,n]}(k_j), \quad P_j(k) = 1, \quad k = k_j, \quad & P_j(k) = 0, \quad k \neq k_j.$$  

We call the Lagrange nodes $k_j = \text{the master wavenumbers (frequencies)}$.

The interpolation order $N$ can be 1, 2, or higher. $N + 1$ system matrices need to be assembled and stored, so one needs to find a trade-off.

The approximated system matrix entries are obtained by multiplying $\tilde{A}_{[m,n]}(k)$ with the inverse of the scaling factor.

**Remark:** The approximated matrix equals the original at $k_j$: $\tilde{A}(k_j) = A(k_j)$.

**Recall:** To avoid assembling the system matrix at each $f$:

- assemble & store matrices @ master frequencies
- perform interpolation @ slave frequencies.
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Determining the frequency windows

**Motivation**: A large polynomial order $N$ required when doing interpolation over the entire frequency band $\Rightarrow$ use small $N$ to obtain smaller intervals.

A few **representative matrix entries** are carefully chosen and assembled at all frequencies. These entries are interpolated simultaneously by an order $N$ polynomial with an a-priori or user-defined accuracy.

**Windows** determined as intervals which contain highest possible number of frequencies in ascending order such that the fitting error for the representative entries inside the interval is below the tolerance.
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Taylor series

\[ x(f) = x(f_0) + x'(f_0)(f - f_0) + \ldots + x^{(q)}(f_0) \frac{(f - f_0)^q}{q!} + \ldots \]
Taylor series for vector functions

**Recall**: We wish to solve $A(f)x(f) = b(f)$ for many $f$.

Notation: $w_{q+1} = \frac{x^{(q)}(f_0)}{q!}$, $A_q = \frac{A^{(q)}(f_0)}{q!}$, $b_q = \frac{b^{(q)}(f_0)}{q!}$.

\[
x(f_0) = A_0^{-1}b_0 = w_1,
\]
\[
x'(f_0) = A_0^{-1}(b_1 - A_1w_1) = w_2,
\]
\[
x^{(q)}(f_0) = A_0^{-1} \left( b_q - \sum_{i=1}^{q} A_i w_{q-i+1} \right) = w_{q+1}.
\]

This moments-computation process is **ill-conditioned**.
A Padé approximant of order \([q_1/q_2]\) of a scalar \(g(f)\) is a rational function

\[
\frac{a_0 + a_1(f - f_0) + \ldots + a_{q_1}(f - f_0)^{q_1}}{1 + b_1(f - f_0) + \ldots + b_{q_2}(f - f_0)^{q_2}},
\]

whose Taylor expansion around \(f_0\) matches the first \(q = q_1 + q_2 + 1\) terms in the Taylor series of \(g(f)\) around \(f_0\).
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**Padé via Asymptotic Waveform Evaluation (AWE)**

Given derivatives of \(g(f)\) up to order \(q\), a linear system with a Hankel matrix is solved for the coefficients \(a_0, \ldots, a_{q_1}\) and \(b_1, \ldots, b_{q_2}\).

For vector functions, such an approximant must be computed for each component of the solution vector \(x(f)\).

Very ill-conditioned and time consuming!
Galerkin Asymptotic Waveform Evaluation

Galerkin AWE forms the moment-matching subspace $W_q = [w_1 \ w_2 \ldots \ w_q]$, imposes that the residual is perpendicular to $W_q$, yielding the following solution vector

$$x_q(f) = W_q \left( W_q^H A(f) W_q \right)^{-1} \left( W_q^H b(f) \right)$$

which matches the solution and the value of $q - 1$ derivatives around $f_0$. 
Galerkin Asymptotic Waveform Evaluation

Galerkin AWE forms the moment-matching subspace \( \mathbf{W}_q = [w_1 w_2 \ldots w_q] \), imposes that the residual is perpendicular to \( \mathbf{W}_q \), yielding the following solution vector

\[
\mathbf{x}_q(\mathbf{f}) = \mathbf{W}_q \left( \mathbf{W}_q^H \mathbf{A}(\mathbf{f}) \mathbf{W}_q \right)^{-1} \left( \mathbf{W}_q^H \mathbf{b}(\mathbf{f}) \right)
\]

which matches the solution and the value of \( q - 1 \) derivatives around \( f_0 \).

Advantages of GAWE:

- a much smaller linear system needs to be solved, namely
  \[
  \left( \mathbf{W}_q^H \mathbf{A}(\mathbf{f}) \mathbf{W}_q \right)^{-1} \left( \mathbf{W}_q^H \mathbf{b}(\mathbf{f}) \right)
  \]
  where \( \mathbf{W}_q^H \mathbf{A}(\mathbf{f}) \mathbf{W}_q \) is of size \( q \times q \)
- yields the Padé approximant of the entire vector \( \mathbf{x}(\mathbf{f}) \).
WCAWE [Slone et al., 2003]

Uses GAWE with the moments computed in a well conditioned manner.
WCAWE [Slone et al., 2003]

Uses GAWE with the moments computed in a well conditioned manner.

Before: \( w_{q+1} = A_0^{-1} \left( b_q - \sum_{i=1}^{q} A_i w_{q-i+1} \right) \).

WCAWE: \( \tilde{w}_{q+1} = A_0^{-1} \left( \sum_{i=1}^{q} b_i c_i - A_1 \tilde{w}_q - \sum_{i=2}^{q} A_i \tilde{w}_{q-i+1} d_i \right) \),

where \( c_i, d_i \) are correction factors.

Moreover, they are orthonormalized via a modified Gram-Schmidt process.
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Proposed algorithm

1. Choose a few representative matrix entries, assemble at all frequencies
2. Apply polynomial interpolation of order \( N \) to scaled entries with deviation \( d_{tol} = 10^{-3} \) ⇒ frequency windows
3. Each frequency window contains \( N + 1 \) master frequencies ⇒ set the highest as the expansion frequency
4. Apply WCAWE inside each window by matching moments at the expansion frequency
   1. Start with a small moment subspace
   2. Add new vectors to the moments subspace as long as residual
      \[
      r(f) = \frac{\|\tilde{A}(f)x_q(f) - b(f)\|_2}{\|b(f)\|_2}
      \]
      is larger than \( \varepsilon_{tol} = 10^{-3} \)
5. Combine spaces from \( N_r = 3 \) windows (multi-point approach)
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Sphere with rigid cap

The pressure outside the sphere verifies:

\[ p(r, \theta) = \frac{-i \rho \nu_0(f)}{2} \sum_{n=0}^{\infty} \left[ \tilde{P}_{n-1}(\cos \alpha) - \tilde{P}_{n+1}(\cos \alpha) \right] \frac{h_n(kr)}{h'_n(ka)} \tilde{P}_n(\cos \theta), \]

\( r \), distance to evaluation point, \( h_n \), spherical Hankel functions of first kind, \( \tilde{P}_n \), Legendre polynomials, \( \nu_0(f) \), uniform normal velocity of spherical cap, and \( a \), radius of sphere. The infinite summation is truncated at \( 2k \).
Parameters for the problem

- sphere radius \( a = 0.6 \, \text{m} \), angle defining the vibrating cap \( \alpha = \pi/3 \, \text{rad} \)
- sound speed is \( c = 340 \, \text{m/s} \), fluid density is \( \rho = 1.225 \, \text{kg/m}^3 \)
- cap normal velocity \( v_0(f) \) is taken as the response of a classical 3 DOF mass-spring-damper: \( M_1 = 60, \ M_2 = 40, \ M_3 = 20 \, \text{(kg)} \); \( K_{1,2,3} = 2.7 \times 10^5 \, \text{(N/m)} \); \( C_{1,2,3} = 20 \, \text{(Ns/m)} \) \( \Rightarrow \) 3 resonances
- mesh with 8 653 nodes, 17 302 triangular elements \( \Rightarrow N_{\text{DOF}} = 15 \, 136 \)
- \( F = [200, 1000] \, \text{Hz} \) with 1 Hz increment (801 individual frequencies)
- \( N = 2 \) interpolation for 8 representative matrix entries \( \Rightarrow 10 \) frequency windows (7 min, 95% on assembly)
Results

FFS vs Matrix Interpolation vs Direct Approach

3h41m12s vs 45h17m23s vs 71h

⇒ speed up factors of 19.4 and 0.6, respectively
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**Car engine compartment**

**Motivation:**
- Vehicles should comply with noise emission regulations
- Engine is a major contributor to vehicle pass by noise
- Acoustic treatments in various locations of engine compartment (e.g., under-bonnet, dash, firewall, floor, etc) are employed
- Interior/exterior acoustics problem: cavity with interior resonances and acoustic radiation in free field.
Parameters for the problem

- Mesh with 9,326 nodes, 18,408 elements ⇒ $N_{DOF} = 10,224$
- 6 field points measured by microphones
- A spherical point source is located at $(x = 3 \, \text{m}, \, y = 7 \, \text{m}, \, z = 0 \, \text{m})$
- $F = [100, 1000] \, \text{Hz}$ with 1 Hz frequency increment (901 frequencies)
- 29 frequency windows
Results

Frequency response at the field point $P_{\text{front}}$ (in front of the engine).

Frequency response at the field point $P_{\text{top}}$ (on top of the engine).

*FFS vs Matrix Interpolation vs Direct Approach*

5h40m59s vs 10h3m1s vs 34h
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MOR for computing FFS for IBEM:

- avoids *assembling and storing* the system matrix at each frequency
- avoids *solving* the linear system at each frequency

**Future work:** Apply similar ideas to MOR 4 MEMS.
Thank you for your attention!