Model order reduction of mechanical systems subjected to moving loads by the approximation of the input

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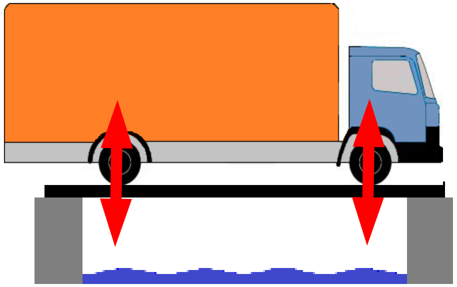
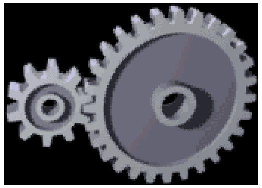
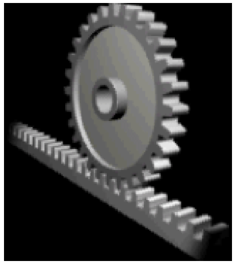
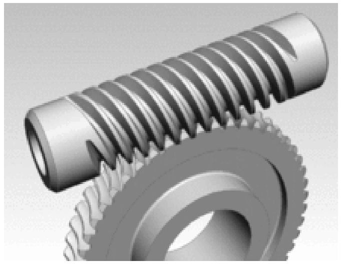
I. Inrtoduction

Elastic multibody systems (EMBS) → ***EMBS with moving loads***

- ***Vehicle-bridge interaction***



- ***Working gears***

www.wikipedia.org

- ***Cableways***


etc.

$$PDEs \xrightarrow{FEM} ODEs$$

Example

$$a_0 \frac{\partial^4}{\partial x^4} w(x,t) + a_1 \frac{\partial^2}{\partial t^2} w(x,t) + a_2 \frac{\partial}{\partial t} w(x,t) = \rho(x,t) u(t)$$

$\downarrow$ FEM

$$b(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \qquad b_i(t) = \int_0^l \rho(x,t) \phi_i(x) dx, \quad i = 1, \ldots, N$$

$\downarrow$

$$\begin{aligned} M\ddot{q}(t) + D\dot{q}(t) + Kq(t) &= b(t)u(t) \\ y(t) &= C(t)q(t) \end{aligned}$$

$$M, D, K \in \mathbb{R}^{N \times N}, \ q \in \mathbb{R}^N, \ b(t) \in \mathbb{R}^N, \ C(t) \in \mathbb{R}^{p \times N}, \ y(t) \in \mathbb{R}^p$$

many forces $\longrightarrow$ input matrix $B(t) \mathbb{R}^{N \times m}$ instead of $b(t)$

High computational cost → *Model order reduction (MOR)*

MOR by projection: $q(t) \approx V\tilde{q}(t), \quad \tilde{q} \in \mathbb{R}^r, \quad r \ll N$

$$\underbrace{W^T M V}_{\tilde{M} \in \mathbb{R}^{r\times r}} \ddot{\tilde{q}}(t) + \underbrace{W^T D V}_{\tilde{D} \in \mathbb{R}^{r\times r}} \dot{\tilde{q}}(t) + \underbrace{W^T K V}_{\tilde{K} \in \mathbb{R}^{r\times r}} \tilde{q}(t) = \underbrace{W^T B(t)}_{\tilde{B}(t) \in \mathbb{R}^{r\times m}} u(t)$$

$$\underbrace{\tilde{y}(t)}_{\tilde{y}(t) \in \mathbb{R}^{p}} = \underbrace{C(t) V}_{\tilde{C}(t) \in \mathbb{R}^{p\times r}} \tilde{q}(t)$$

Systems with time-varying input and/or output matrices:

$V, W -$ **???**

II. Approximation of the input matrix

$\rho(x,t)=g(x-\zeta(t))$, $\quad \zeta(t)$ *- a position of a «centre of force» at t*

$$\zeta(t)\in\Omega\subseteq[0,l]$$

$$b(t)=\begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \qquad b_i(t)=\int_0^l g(x-\zeta(t))\phi_i(x)dx, \quad i=1,\dots,N$$

Consider a SISO system

$$\begin{aligned} M\ddot{q}(t)+D\dot{q}(t)+Kq(t) &= b(t)u(t) \\ y(t) &= b^T(t)q(t) \end{aligned}$$

with $t\in[0,T]$

Naive approach:

$$\begin{aligned} M\ddot{q}(t)+D\dot{q}(t)+Kq(t) &= I(b(t)u(t)) \\ y(t) &= b^T(t)q(t) \end{aligned}$$

Difficulty: many inputs

II. Approximation of the input matrix

Goal: approximate $b(t)$ in a lower dimension subspace

$$b(t) \approx \hat{B}\,\chi(\zeta(t)) = \sum_{i=1}^{n} \hat{b}_i \chi_i(\zeta(t)), \quad n \ll N$$

$$\begin{aligned} M\ddot{\hat{q}}(t) + D\dot{\hat{q}}(t) + K\hat{q}(t) &= \hat{B}\hat{u}(t) \quad \textit{with} \quad \hat{u}(t) = \chi(\zeta(t))\,u(t) \\ \hat{y}(t) &= \hat{B}^T\hat{q}(t) \end{aligned}$$

Note: $y(t) \approx \chi(\zeta(t))^T \hat{y}(t)$

Error bound:

$$\left\| \begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix} - \begin{bmatrix} \hat{q}(t) \\ \dot{\hat{q}}(t) \end{bmatrix} \right\|_\infty \leqslant \eta \left\| b - \hat{B}\,\chi \right\|_\infty$$

Two approximation approaches:

1. *given the matrix* $\hat{B}$, *find the vector* $\chi(\zeta)$
2. *given the vector* $\chi(\zeta)$, *find the matrix* $\hat{B}$

such that $\left\| b - \hat{B}\,\chi \right\|_\infty \to \min$

II. Approximation of the input matrix

$$b(t)=\begin{bmatrix}\varphi_1(\zeta(t))\\ \varphi_2(\zeta(t))\\ \vdots\\ \varphi_N(\zeta(t))\end{bmatrix}\approx\begin{bmatrix}\hat{b}_{11} & \cdots & \hat{b}_{1n}\\ \hat{b}_{21} & \cdots & b_{2n}\\ \vdots & \ddots & \vdots\\ \hat{b}_{N1} & \cdots & \hat{b}_{Nn}\end{bmatrix}\begin{bmatrix}\chi_1(\zeta(t))\\ \vdots\\ \chi_n(\zeta(t))\end{bmatrix}=\hat{B}\chi(\zeta(t))$$

- *approximation by polynomial expansion* $\varphi_i(x)\approx\sum_{j=1}^{n}\hat{b}_{ij}P_{j-1}(x),\quad i=1,\dots,N,$

  where $P_0(x),\dots,P_{n-1}(x)$ *are orthogonal polynomials;*

- *B-spline interpolation* $\varphi_i(x)\approx\sum_{j=1}^{n}\hat{b}_{ij}\beta_{j-2}(x),\quad i=1,\dots,N$

  where $\beta_{-1}(x),\dots,\beta_{n-2}(x)$ *are B-splines;*

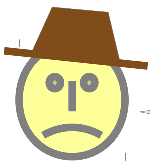
- *linear least square method (LLSM)* $\varphi_i(x)=\varphi_i^{(N)}(x)\approx\sum_{j=1}^{n}\hat{b}_{ij}\phi_j^{(n)}(x),\quad i=1,\dots,N$

  where $\phi_1^{(n)}(x),\dots,\phi_n^{(n)}(x)$ *are FEM basis functions on a coarse grid;*

- *empirical interpolation method (EIM)*
  [Barrault, Maday, Nguyen, Patera, 2004]

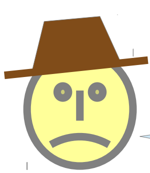
III. Model order reduction of mechanical systems

- *Balanced truncation*
  - *solving Lyapunov equations is required*
    - *use SO-LR-ADI method specially adapted for second-order systems* [Benner, Kürschner, Saak, 2012]



But, for mechanical systems with a weak damping, this method converges very slowly

- *Krylov subspace methods*
  - *SOAR* [Bai, Su, 2005; Salimbahrami, 2005]
  - *SOR-IRKA, SO-IRKA* [Wyatt, 2012]
  - *AORA* [Lee, Chu, Feng, 2004; Bodendiek, Bollhöfer, 2013]
  - *MIRKA* [Soppa, 2011]



Choice of interpolation points and directions for second-order systems is still unclear

Our approach: subspace acceleration poles finding combined with an extention of a frequency range

IV. Numerical experiments

Test model with a moving load: 1D Euler-Bernoulli beam equation

$$\rho A \frac{\partial^2}{\partial t^2} w(x,t) + 2\rho A \omega_d \frac{\partial}{\partial t} w(x,t) + EI \frac{\partial^4}{\partial x^4} w(x,t) = \delta(x - vt) u$$

$(x,t) \in (0,l) \times (0,T)$ ***(has an analytical solution)*** [Fryba, 1999]

$w(x,t)$ *is a vertical deflection of the beam*

$\delta(x-\zeta(t))$ *is a point force density*

v *is a velocity of the moving load*

$\zeta(t) = vt$ *is an instantaneous position of a force*

u *is a magnitude of the moving load*

ρ *is a mass density*

A *is a cross section area*

ω_d *is a circular frequency of damping*

E *is an Young modulus*

I *is an area moment of inertia*

with simply supported ends of the beam

$$w(0,t) = 0, \quad \frac{\partial^2}{\partial x^2} w(0,t) = 0,$$

$$w(l,t) = 0, \quad \frac{\partial^2}{\partial x^2} w(l,t) = 0$$

and initial conditions

$$w(x,0) = 0, \quad \frac{\partial}{\partial t} w(x,0) = 0$$

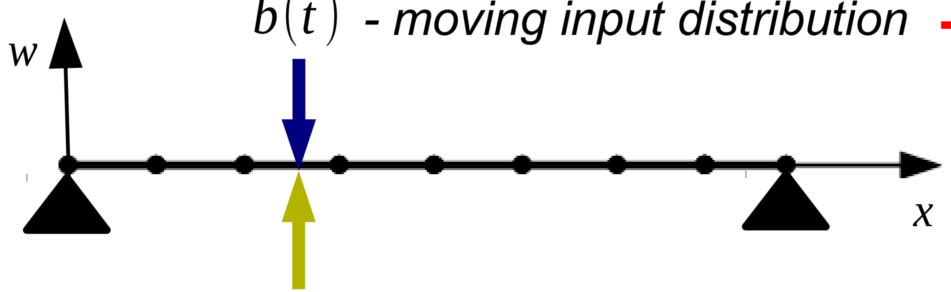
input distribution vector

$$b(t)=\begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_N(t) \end{bmatrix}, \quad b_i(t)=\int_0^l \delta(x-\zeta(t))\phi_i(x)\,dx=\phi_i(\zeta(t)), \quad i=1,..,N,$$

where $\phi_i(x)$ *is a finite element method basis function corresponded to some node and* $\zeta(t)\in\Omega=[0,l]$

$$\begin{aligned} M\ddot{q}(t)+D\dot{q}(t)+Kq(t) &= b(t)u \\ y(t) &= b^T(t)q(t) \end{aligned}$$

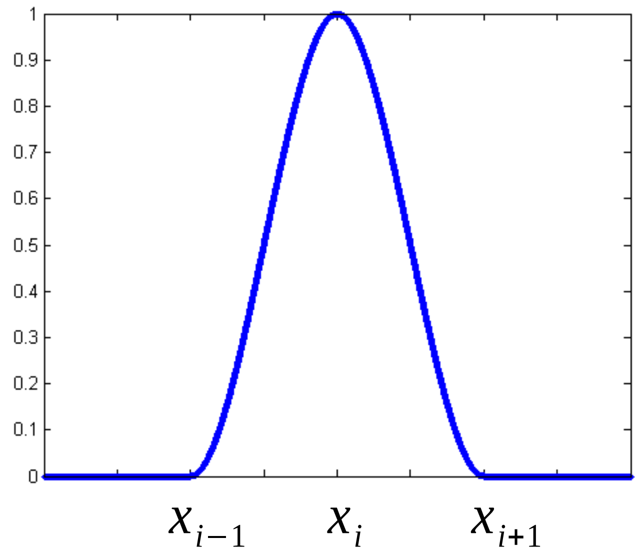
$b(t)$ *- moving input distribution* → $b(t)u$ *- moving load*

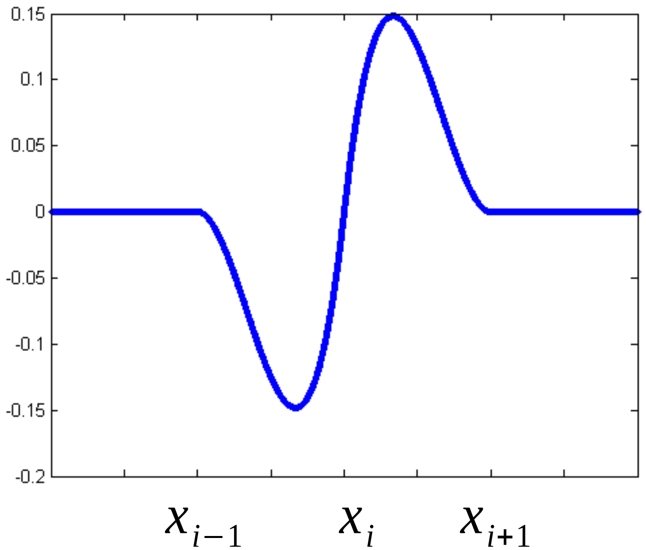


$c(t)=b^T(t)$ *- moving output distribution* → $y(t)$ *- moving observation*

Approximation of FEM basis functions

$$\phi_i(\zeta(t)) \approx \sum_{j=1}^{n} \hat{b}_{ij}\chi_j(t), \quad j=1,\dots,N, \quad n \ll N$$

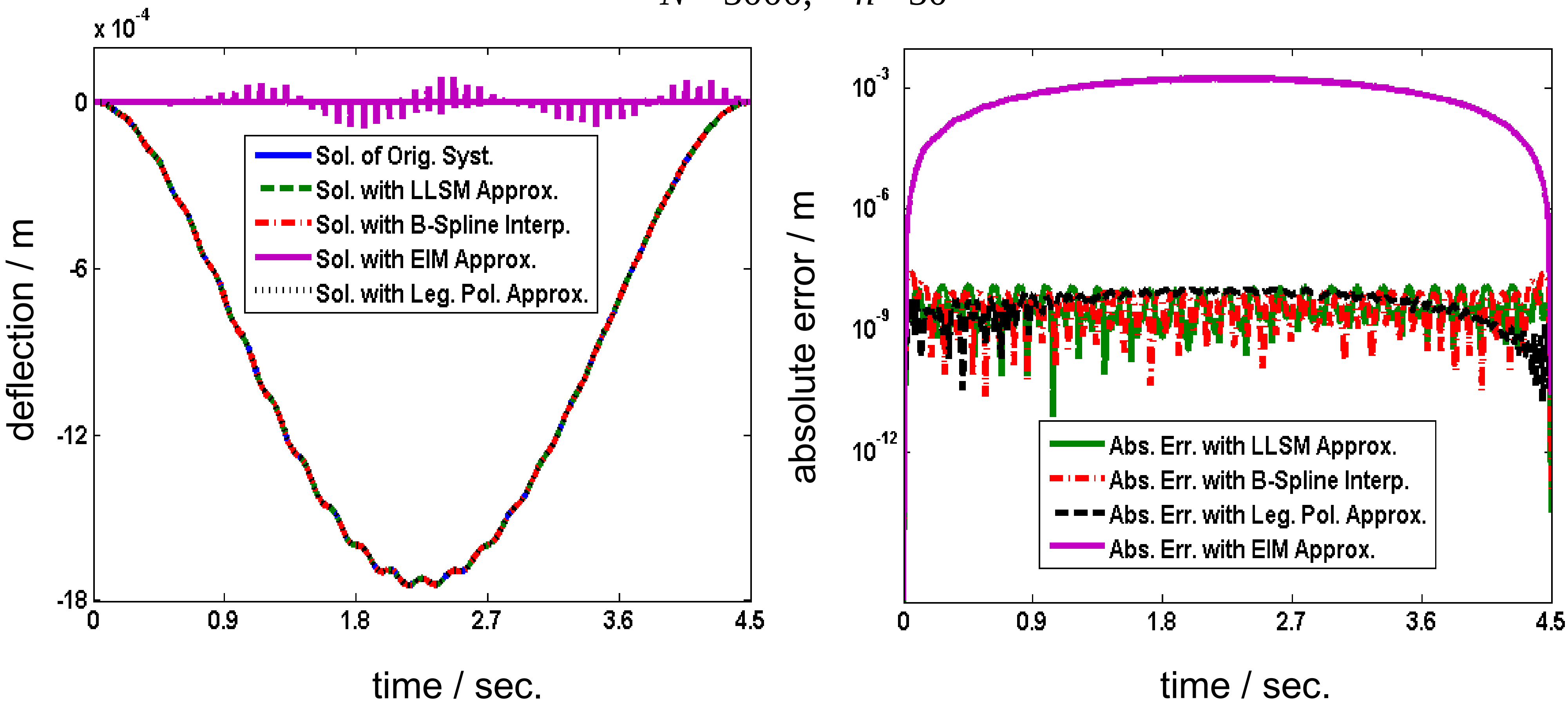




FEM basis functions $\phi_i(x)$

Approximated output by approximations of the input

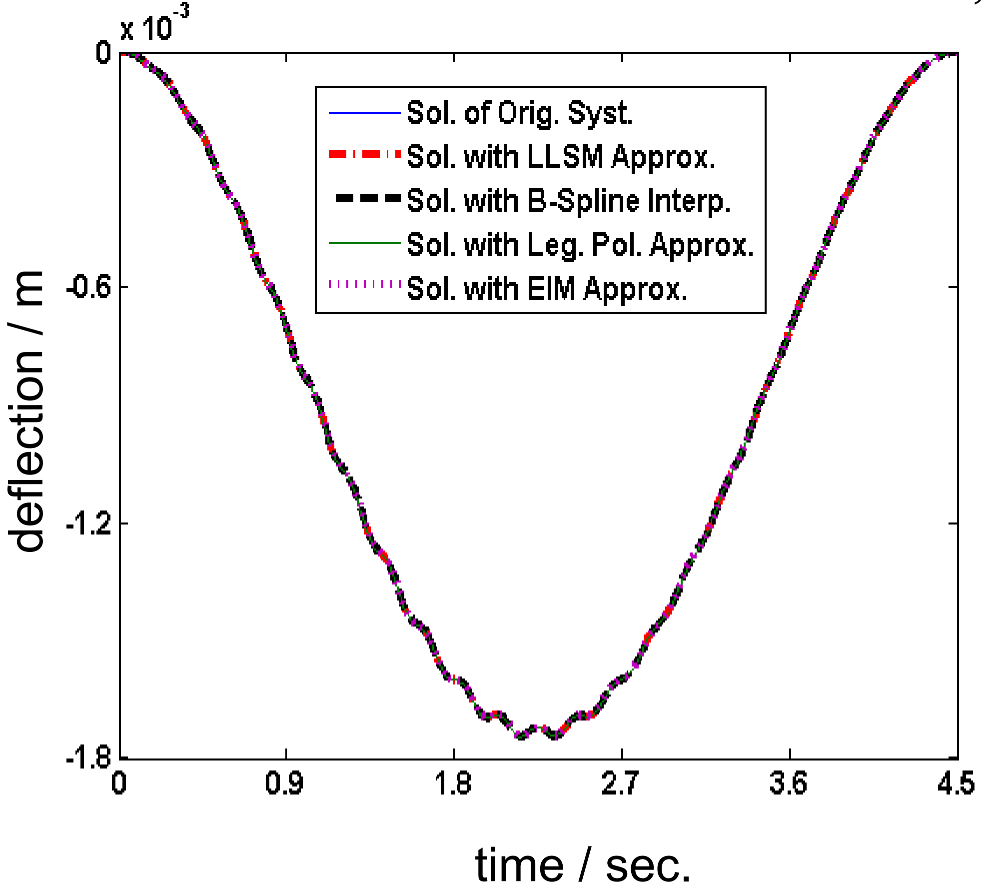
$N=5000, \quad n=50$

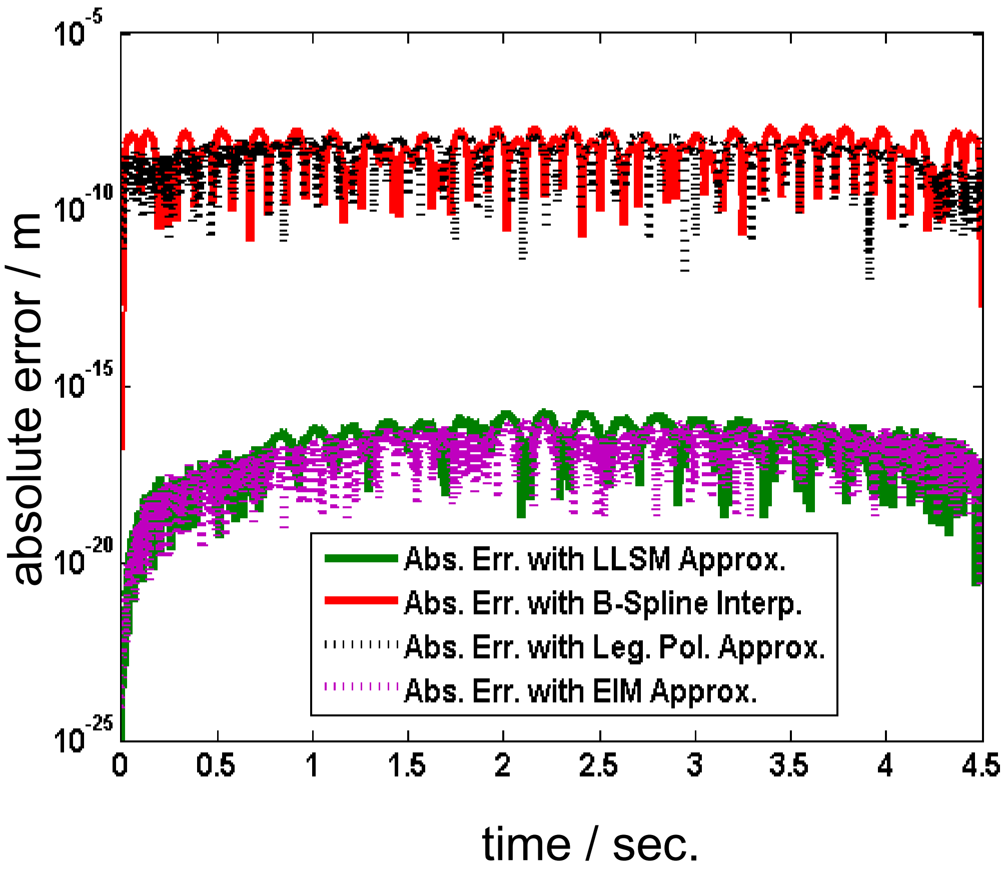


$$\phi_i(\zeta(t)) \approx \sum_{j=1}^{50} \hat{b}_{ij}\, \chi_j(t), \quad i=1,\dots,5000$$

Approximated output by approximations of the input

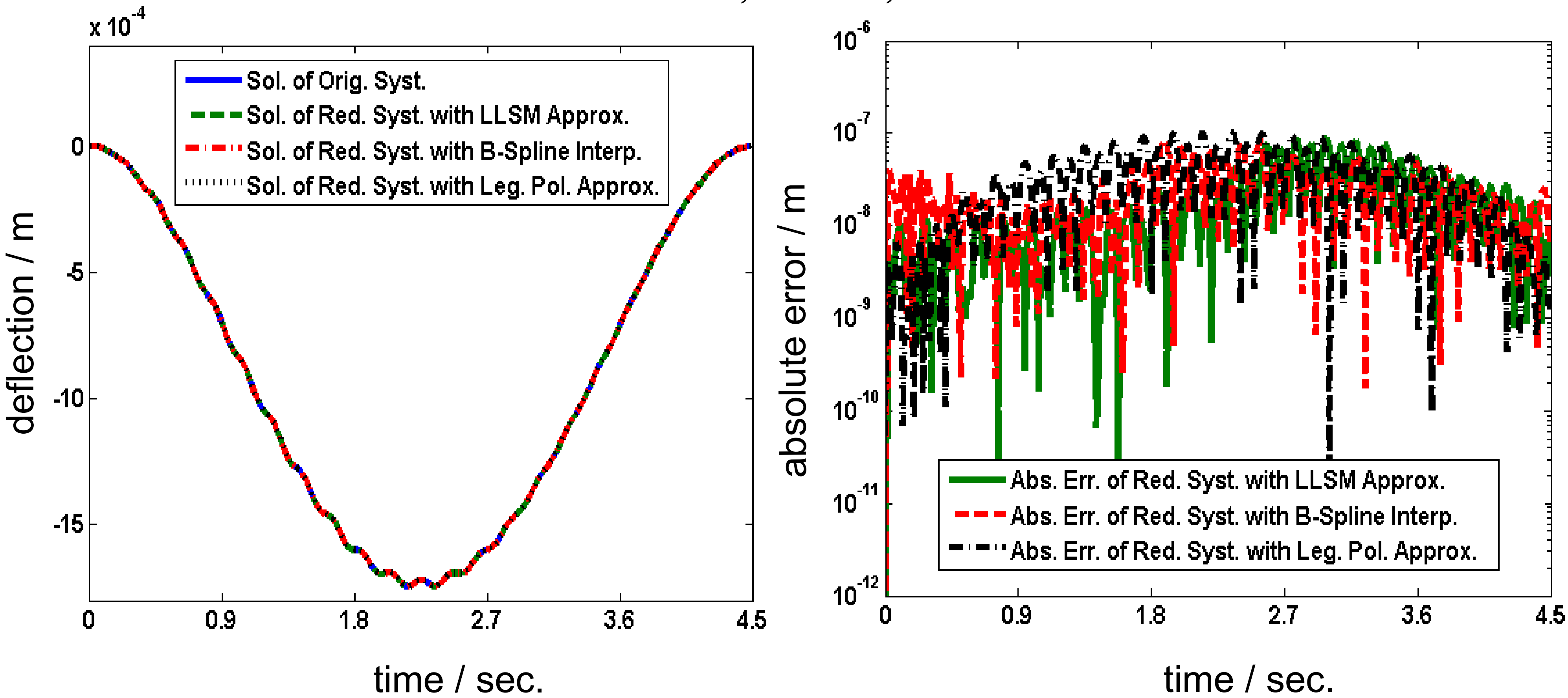
$N=50, \quad n=50$





Approximated output of the reduced system with moving load

$N=5000, \quad n=50, \quad r=20$



Reduction is carried out by the software `MatMorembs`

http://www.itm.uni-stuttgart.de/research/model_reduction/MOREMBS_MatMorembs_en.php

Eberhard, Lehner, Fehr, Nowakowski, Fischer, Kürschner et al.

V. Conclusion

It was considered:

- *second-order systems with moving load*
- *approximation of time-varying input matrix*
- *model reduction methods for mechanical systems*

Further work:

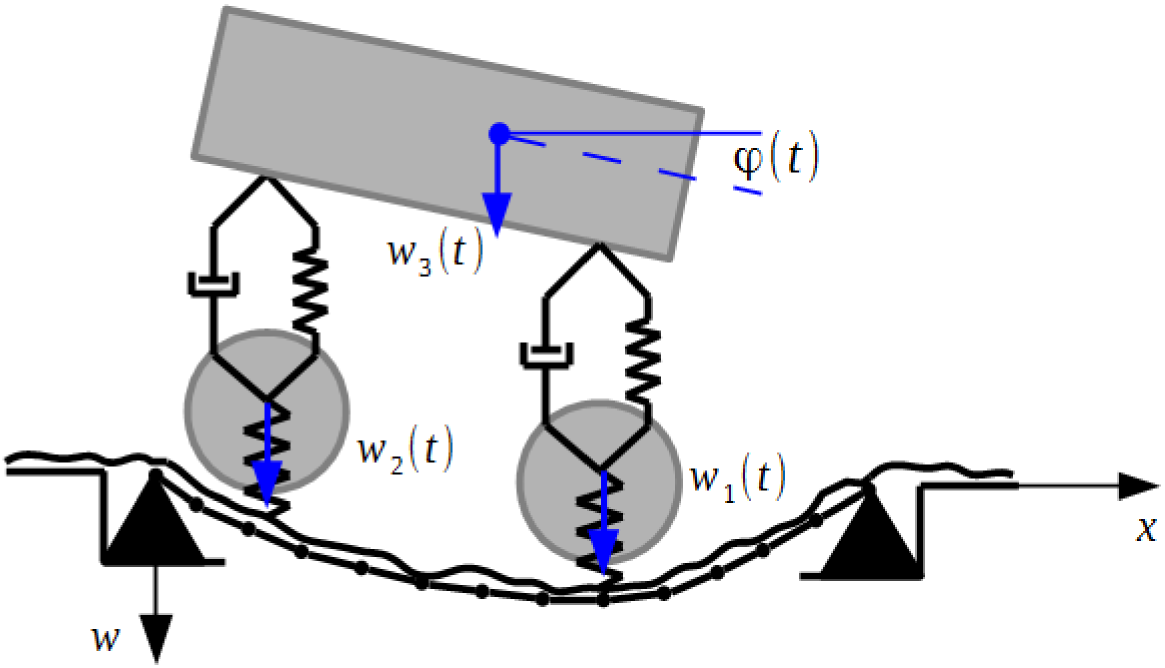
- *search of optimal methods to reduction of mechanical systems*
- *search of new approaches to model reduction of systems with moving load*

Further work:

- *consideration of more realistic models*

for example, a coupled bridge-vehicle system

beam subjected to a moving two-axle system



Thank you for your attention!