Recent advances in the reduction of frequency based models for structures and vibrations

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December 11–13, 2013

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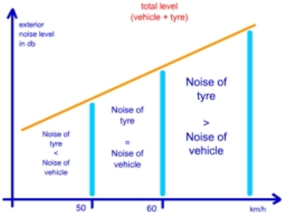
Outline

Outline

Examples structures and vibrations

Car tyres

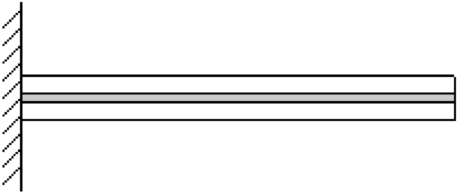


Windscreens

- Structural damping
- Choice of connection (glue) to the car

Planes

Nonlinear damping



- Clamped sandwhich beam
- 168 degrees of freedom (finite elements)
- Linear system

$$(K_e + \frac{G_0 + G_\infty (i\omega\tau)^\alpha}{1 + (i\omega\tau)^\alpha} K_v - \omega^2 M)x = f$$

with $\alpha = 0.675$ and $\tau = 8.230$.
Parameters obtained from measurements.

Lamot footbridge damper optimization

- Lamot bridge finite element model ($n = 25,962$)
- The goal is to determine the optimal stiffness and damping coefficient of four bridge dampers (=8 parameters).

Objective

- Large dynamical system in the frequency domain:

$$\begin{aligned} A_n(\omega)x &= f \cdot u(\omega) \\ y &= c^T x \end{aligned}$$

  with $x \in \mathbb{C}^n$ with n large.
  We call $H = y/u$ the *transfer function* or *frequency response function*.
- Reduce to

$$\begin{aligned} A_r(\omega)\widetilde{x} &= \widetilde{f} \cdot u(\omega) \\ \widetilde{y} &= \widetilde{c}^T \widetilde{x} \end{aligned}$$

  with $\widetilde{x} \in \mathbb{C}^r$ with $r \ll n$
- Subspace methods:
  - Right subspace $\mathrm{Range}(V)$ with $V \in \mathbb{C}^{n\times r}$ (control side)
  - Left subspace $\mathrm{Range}(W)$ with $W \in \mathbb{C}^{n\times r}$ (observation side)

$$\begin{aligned} (W^*AV)\widetilde{x} &= (W^*f) \\ \widehat{y} &= (V^*c)^T \widetilde{x} \end{aligned}$$

Objective

- Krylov methods
- Dominant pole algorithm

both for

- Nonlinear frequency dependence

$$\begin{aligned} (A_0 + i\omega A_1 + e^{-i\omega\tau} A_2)x &= f \\ y &= c^T x \end{aligned}$$

- Parametric models

$$\begin{aligned} (A_0 + i\omega A_1 + \gamma_1 A_2 + i\omega\gamma_2 A_3)x &= f \\ y &= c^T x \end{aligned}$$

Outline

Polynomial models

- Polynomial model

$$\begin{aligned}(A_0 + sA_1 + \cdots + A_N s^N)x &= f \\ y &= c^T x\end{aligned}$$

- Companion linearization

$$\begin{aligned}(\mathcal{A} + s\mathcal{B})\mathbf{x} &= \mathbf{b} \\ y &= (e_1 \otimes c)^T \mathbf{x}\end{aligned}$$

with

$$\mathcal{A}+s\mathcal{B} = \begin{bmatrix} A_0 + sA_1 & sA_2 & \cdots & sA_N \\ -sI & I & & 0 \\ & \ddots & \ddots & \vdots \\ & & -sI & I \end{bmatrix}, \mathbf{x} = \begin{pmatrix} x \\ sx \\ \vdots \\ s^{N-1}x \end{pmatrix}, \mathbf{b} = \begin{pmatrix} f \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Nonlinear models

- General form:

$$\begin{aligned} A(s)x &= f \\ y &= c^T x \end{aligned}$$

- Approximate by a polynomial or rational polynomial by interpolation in $N+1$ points:

$$\sigma_0, \sigma_1, \ldots, \sigma_N, \ldots$$

  by a (rational) polynomial

Nonlinear models

- Different bases:
  - Newton polynomials:

$$N_{j+1}(\lambda) = N_j(\lambda) \cdot (\lambda - \sigma_j) \quad j = 1, 2, \ldots$$

  - Rational Newton polynomials

$$(\lambda - \xi_j) \cdot N_{j+1}(\lambda) = (\lambda - \sigma_j) \cdot N_j(\lambda)$$

  - barycentric Lagrange polynomials, ...
- Variants
  - Dynamic: choose polynomial coefficients and degree during the iterations
  - Static: fix the polynomial and then reduce

Newton polynomial basis

- Newton polynomials:

$$N_0 := 1, N_1 := (\lambda - \sigma_0), N_2 := (\lambda - \sigma_0)(\lambda - \sigma_1), \ldots$$

$$N_{j+1}(\lambda) = N_j(\lambda) \cdot (\lambda - \sigma_j) \quad j = 1, 2, \ldots$$

- $A(s) = A_0 N_0 + A_1 N_1 + \cdots$ with
- $\sigma_0, \sigma_1, \ldots$ are interpolation points.
- Linearization:

$$\left(\begin{bmatrix} A_0 & A_1 & \cdots & A_N \\ \sigma_0 I & I & & \\ & \ddots & \ddots & \\ & & \sigma_{N-2} I & I \end{bmatrix} - s \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ N_1(s)x \\ N_2(s)x \\ \vdots \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Linear reduced model

- Build reduced model for the linearization. Vectors:

$$\mathbf{V} = \begin{pmatrix} V \\ \star \\ \vdots \\ \star \end{pmatrix} \qquad \mathbf{W} = \begin{pmatrix} W \\ \star \\ \vdots \\ \star \end{pmatrix}$$

- Reduced model for the linearization:

$$\begin{aligned} (\widetilde{A} + s\widetilde{B})z &= \widetilde{f} \\ \widetilde{H} &= \widetilde{c}^T z \end{aligned}$$

  where $\widetilde{A} = \mathbf{W}^* A_N \mathbf{V}$, $\widetilde{B} = \mathbf{W}^* B_N \mathbf{V}$.
- For example, a delay differential equation can then be solved by an ODE solver
- [Freund 2005], [Michiels, Jarlebring, M. 2012]

Properties of linearized model

- Original equation:

$$A(\lambda)x = f$$

- Linearized equation ($N = 2$)

$$\left(\begin{bmatrix} A_0 & A_1 & A_2 \\ \sigma_0 I & I & \\ & \sigma_1 I & I \end{bmatrix} - s \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ N_1(s)x \\ N_2(s)x \end{bmatrix} = \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

- Eigenvectors

$$\begin{aligned} A(\lambda)p &= 0 \\ \begin{bmatrix} A_0 & A_1 & A_2 \\ (\sigma_0 - \lambda)I & I & \\ & (\sigma_1 - \lambda)I & I \end{bmatrix} \begin{pmatrix} p \\ N_1(\lambda)p \\ N_2(\lambda)p \end{pmatrix} &= 0 \end{aligned}$$

- Left eigenvectors

$$\begin{aligned} q^* A(\lambda) &= 0 \\ \begin{pmatrix} q \\ -A_1^* q + (\sigma_1 - \lambda) A_2^* q \\ -A_2^* q \end{pmatrix}^* \begin{bmatrix} A_0 & A_1 & A_2 \\ (\sigma_0 - \lambda)I & I & \\ & (\sigma_1 - \lambda)I & I \end{bmatrix} &= 0 \end{aligned}$$

Properties of linearized model

- Moments for simple interpolation points:

$$\sigma_0, \sigma_1, \sigma_2$$

- Nonlinear problems has moments: $m_j = A(\sigma_j)^{-1} f$ for $j = 0, 1, 2$.
- Linearization of the polynomial has moments:

$$\mathbf{m}_0 = \begin{pmatrix} m_0 \\ 0 \\ 0 \end{pmatrix} \quad \mathbf{m}_1 = \begin{pmatrix} m_1 \\ N_1(\sigma_1)m_1 \\ 0 \end{pmatrix} \quad \mathbf{m}_2 = \begin{pmatrix} m_2 \\ N_1(\sigma_2)m_2 \\ N_2(\sigma_2)m_2 \end{pmatrix}$$

- For 2-sided moment matching, see [Michiels, Peeters 2013]

Nonlinear reduced model

- Original matrix:

$$A(s) = \sum_{j=0}^{m} f_j(s) A_j$$

- Krylov vectors V and W of linearization
- Reduced nonlinear model:

$$\begin{aligned} \left(\sum_{j=0}^{m} f_j(s)\widetilde{A}_j\right) z &= \widetilde{f} \\ \widetilde{H} &= \widetilde{c}^T z \end{aligned}$$

  where $\widetilde{A}_j = W^* A_j V$.
- SOAR [Bai, Su 2005]

Algorithms

- Apply a Krylov method to the linearization (moment matching)
  - Krylov methods: two-sided Arnoldi
  - Rational Krylov methods: the poles correspond to the interpolation points σ_j
- Use the dominant pole algorithm:
  - The σ_j are the Ritz values computed by the dominant pole algorithm
  - Linearization supports deflation (which is needed to compute more than one pole)

Rational Krylov method

- Linear system:

$$\begin{aligned} (A+sB)x &= f \\ y &= c^T x \end{aligned}$$

- Two-sided model reduction
  - Interpolation in $s_0, \ldots, s_N$:

$$\begin{aligned} V &= \mathrm{orth}((A+s_0B)^{-1}f, \ldots, (A+s_NB)^{-1}f) \\ W &= \mathrm{orth}((A+s_0B)^{-*}c, \ldots, (A+s_NB)^{-*}c) \end{aligned}$$

  - Moment matching property:

$$\widetilde{H}(s_j) = H(s_j)$$

Rational Krylov method

- One sided algorithm:

  Let $b = (A + \sigma_0 B)^{-1} f$
  Let $v_1 = b/\|b\|$
  for $j = 1, \ldots, k$ **do**
  Solve
  $$(A + \sigma_j B)t_j = Bv_j$$
  Orthonormalize t_j against V_j and add: V_{j+1}
  end for

- Numerically more stable than accumulating moments.

Rational Krylov

Shift-and-invert step: $(A - \sigma_j B)w = Bv$

1. First step: multiply by B: shift vector downwards

$$
\begin{array}{ccc} B \cdot v & = & w \end{array}
$$

$$
\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \cdot \begin{bmatrix} \star \\ \star \\ \vdots \\ \star \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \star \\ \star \\ \vdots \\ \star \end{bmatrix}
$$

2. Then solve with A

$$
\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \sigma_0 I & I & & & & \\ & \sigma_1 I & I & & & \\ & & \sigma_2 I & I & & \\ & & & \sigma_3 I & I & \\ & & & & \sigma_4 I & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \star \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ \star \end{bmatrix}
$$

This is rational Krylov with poles $\sigma_1, \sigma_2, \ldots$

Rational Krylov

Shift-and-invert step: $(A - \sigma_j B)w = Bv$

1. First step: multiply by B: shift vector downwards

$$B \cdot v \quad = \quad w$$

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \cdot \begin{bmatrix} \star \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \star \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

2. Then solve with $A - \sigma_1 B$

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \star & I & & & & \\ & 0 & I & & & \\ & & \star & I & & \\ & & & \star & I & \\ & & & & \star & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \star \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is rational Krylov with poles $\sigma_1, \sigma_2, \ldots$

Rational Krylov

Shift-and-invert step: $(A - \sigma_j B)w = Bv$

1. First step: multiply by B: shift vector downwards

$$B \cdot v \quad = \quad w$$

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \cdot \begin{bmatrix} \star \\ \star \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \star \\ \star \\ 0 \\ 0 \end{bmatrix}$$

2. Then solve with $A - \sigma_2 B$

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \star & I & & & & \\ & \star & I & & & \\ & & 0 & I & & \\ & & & \star & I & \\ & & & & \star & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \star \\ \star \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ \star \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This is rational Krylov with poles $\sigma_1, \sigma_2, \ldots$

Rational Krylov

Shift-and-invert step: $(A - \sigma_j B)w = Bv$

1. First step: multiply by B: shift vector downwards

$$B \cdot v = w$$

$$\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \cdot \begin{bmatrix} \star \\ \star \\ \star \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \star \\ \star \\ \star \\ 0 \end{bmatrix}$$

2. Then solve with $A - \sigma_3 B$

$$\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \star & I & & & & \\ & \star & I & & & \\ & & \star & I & & \\ & & & 0 & I & \\ & & & & \star & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \star \\ \star \\ \star \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ 0 \\ 0 \end{bmatrix}$$

This is rational Krylov with poles $\sigma_1, \sigma_2, \ldots$

Rational Krylov

Shift-and-invert step: $(A - \sigma_j B)w = Bv$

1. First step: multiply by B: shift vector downwards

$$
\begin{array}{ccc} B \cdot v & = & w \end{array}
$$

$$
\begin{bmatrix} 0 & \cdots & \cdots & 0 \\ I & & & \vdots \\ & \ddots & & \vdots \\ & & I & 0 \end{bmatrix} \cdot \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \star \\ \star \\ \star \\ \star \end{bmatrix}
$$

2. Then solve with $A - \sigma_4 B$

$$
\begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & A_5 \\ \star & I & & & & \\ & \star & I & & & \\ & & \star & I & & \\ & & & \star & I & \\ & & & & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \star \\ \star \\ \star \\ \star \\ 0 \end{bmatrix} = \begin{bmatrix} \star \\ \star \\ \star \\ \star \\ \star \\ 0 \end{bmatrix}
$$

This is rational Krylov with poles $\sigma_1, \sigma_2, \ldots$

Numerical example

'gun' problem Manchester-Berlin NEVP collection

$$\begin{cases} A(\omega)x(\omega) & = & fu(\omega) \\ y(\omega) & = & cx(\omega) \end{cases},$$

where

$$A(\omega) = K - \omega^2 M + i\sqrt{\omega^2 - \sigma_1^2}W_1 + i\sqrt{\omega^2 - \sigma_2^2}W_2,$$

with

- M, K, W_1, W_2 real symmetric matrices of dimension 9956.
- K positive semidefinite and M positive definite.
- W_1 and W_2 low rank matrices.
- The vectors f and c are

$$f = [1\ 1\ \cdots\ 1]^T/\sqrt{n}, \qquad c = [1\ 1\ \cdots\ 1]^T/\sqrt{n}.$$

Example: 'gun' problem

Reduced order models were constructed

$$\begin{cases} \widehat{A}(\omega)x(\omega) & = & \widehat{f}u(\omega) \\ y(\omega) & = & \widehat{c}^*\widehat{x}(\omega) \end{cases},$$

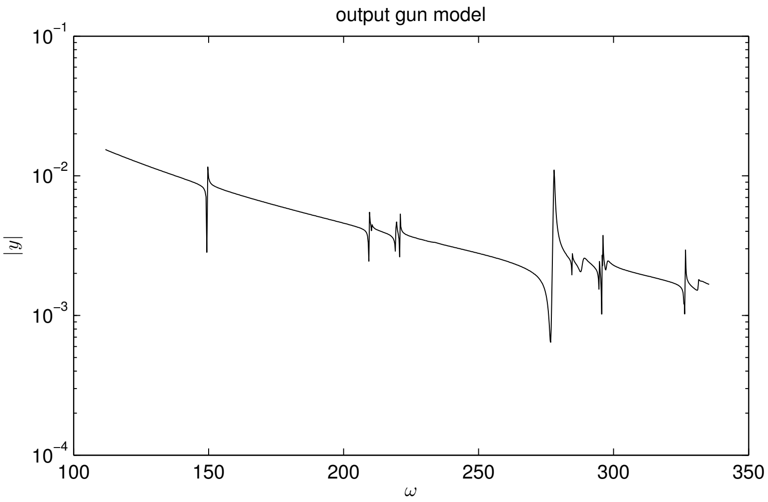
where $\widehat{x}(\omega) \in \mathbb{C}^k$ and

$$\widehat{A}(\omega) = V_k^* K V_k - \omega^2 \; V_k^* M V_k + i\sqrt{\omega^2 - \sigma_1^2} \; V_k^* W_1 V_k$$
$$+ \; i\sqrt{\omega^2 - \sigma_2^2} \; V_k^* W_2 V_k,$$

$$\widehat{f} = V_k^* f,$$

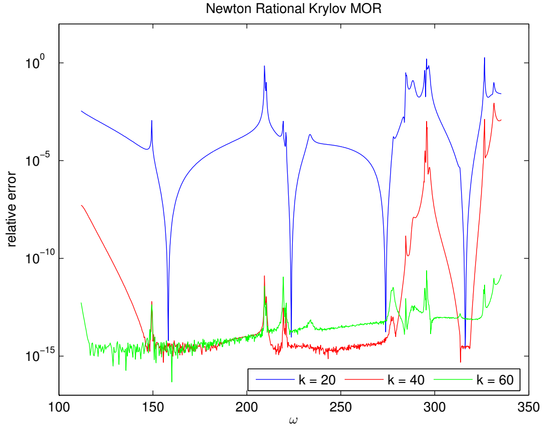
$$\widehat{c} = V_k^* c$$

Example: ‘gun’ problem



Example: 'gun' problem

Hermite interpolation in 4 points:



reduced model	full model
3.8 s ($k = 20$) 4.9 s ($k = 40$) 7.1 s ($k = 60$)	512 s

Example: boundary element problem

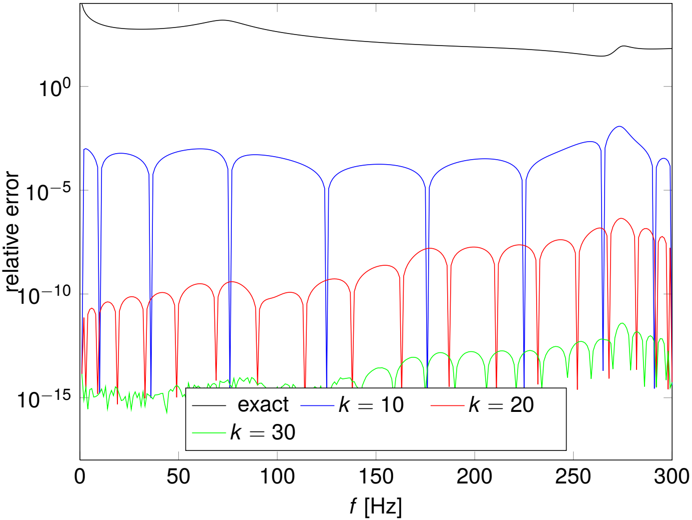
Boundary element formulation for the Helmholtz equation (LMS — Siemens):

- Nonlinear in ω
- Full matrices

Procedure as follows:

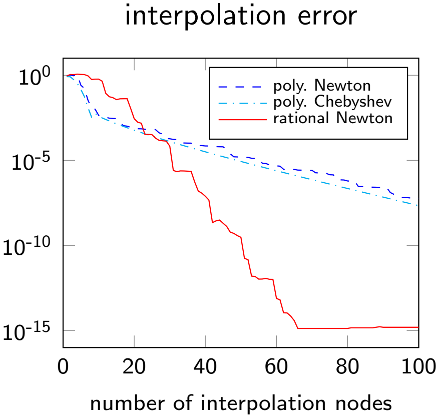
1. Interpolate in k points on the ω axis
2. Build the interpolating polynomial (barycentric Lagrange polynomials)
3. Build reduced polynomial problem of degree $k-1$ with matrices of dimension $k \times k$.

Example: boundary element problem



Other approximations

- [Michiels, Jarlebring and M. 2012] Use spectral discretization for delay equation
- [Güttel, Van Beeumen, M., Michiels 2013] Use rational Newton polynomials



Dominant pole algorithm

- Computes poles of

$$c^*A(s)^{-1}f \approx \sum_j \frac{R_j}{s-\lambda_j}$$

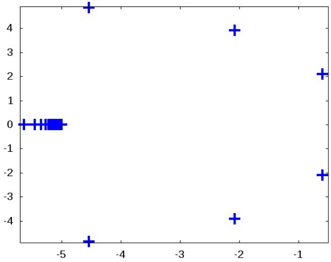
with

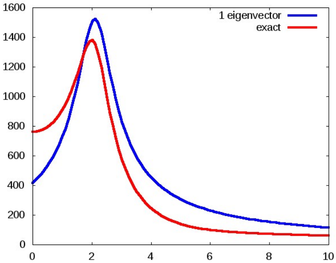
$$R_j = \frac{(q_j^* f)(c^* p_j)}{q_j^* A'(\lambda_j) p_j}$$

and $A(\lambda_j)p_j = 0$ and $q_j^* A(\lambda_j) = 0$.

- DPA = Newton's method applied to $1/(c^*A(s)^{-1}f)$.
- It converges to the eigenvalue with largest

$$\frac{|R_j|}{|\mathrm{Re}(\lambda_j)|}$$





Dominant pole algorithm

- For a linear problem, DPA with subspace acceleration is like rational Krylov/Jacobi-Davidson for the eigenvalue problem:
  1. solve a sequence of linear systems:

  $$\begin{aligned} (A + s_j B)x_j &= f \\ (A + s_j B)^* t_j &= c \end{aligned}$$

  2. Accumulate in subspace: $V_j = [V_{j-1}, x_j]$, $W_j = [W_{j-1}, t_j]$.
  3. Solve projected eigenvalue problem

  $$\det((W_j^* A V_j) + \lambda (W_j^* B V_j)) = 0 \quad \rightarrow \quad \text{new } s_{j+1}$$

  (= two-sided version of rational Krylov)
- For computing more than one eigenvalue, modify the definition of dominance:

  $$f - (Bp)(q^* f)$$

  [Rommes, & Martins 2006, 2008]

Dominant pole algorithm

- For a nonlinear problem, we do the same, but for a polynomial eigenvalue problem interpolating in the s_j's.
  1. solve a sequence of linear systems:

$$\begin{aligned} A(s_j)x_j &= f \\ A(s_j)^* y_j &= c \end{aligned}$$

  2. Solve projected nonlinear eigenvalue problem

$$\det(W_j^* A(\lambda) V_j) = 0 \quad \rightarrow \quad \text{new } s_{j+1}$$

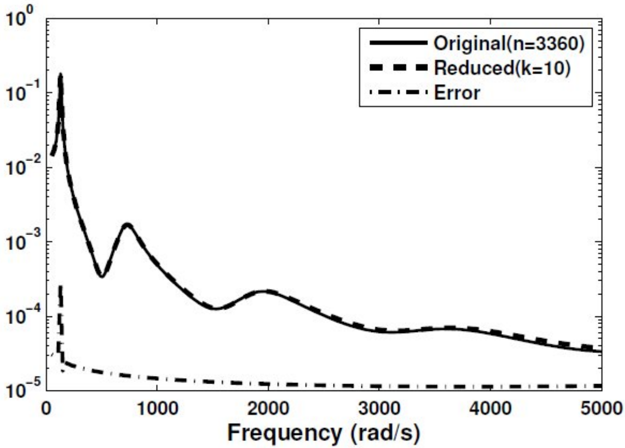
- Deflation can be reformulated as:

$$f - \frac{A(s)p}{(\lambda - s)(q^* A'(\lambda)p)} q^* f$$

(Follows from the use of a 'linearization')
[Saadvandi, M. & Jarlebring 2012]

Example: Sandwhich beam

- Computed five dominant poles
- Small scale (nonlinear) eigenvalue problem is solved with rational Krylov



Outline

Parametric models

- Model:

$$\left((A_0 + \sum_{j=1}^{p} \gamma_j A_j) + s(B_0 + \sum_{j=1}^{p} \gamma_j B_j) \right) x = f$$
$$y = c^T x$$

  with $x \in \mathbb{C}^n$ with n large; $\gamma \in \mathbb{R}^p$ are parameters
- Examples:
  - The mean of the output:

$$z = \int_\Gamma y d\gamma$$

    Numerical integration using cubature rule:

$$z \approx \sum_{j=1}^{N} w_j \cdot y(\gamma_j)$$

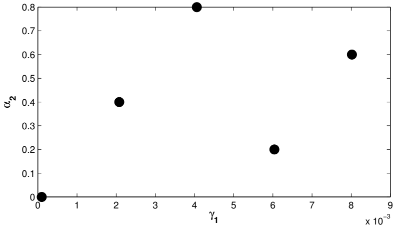
  - Optimization problem:

$$\min_{\gamma \in \Gamma} |y|^2$$

Discretize the parameter space

Discretization of parameter space:

- Cartesian grid (only for small number of parameters)
- Sparse grids
- Lattice rules



$$1, 1, 2, 3, 5, 8, \ldots, Z, N\ .$$

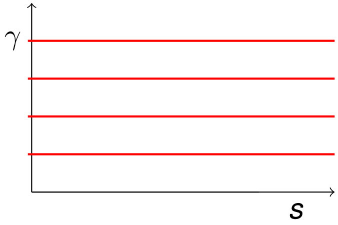
We define the N lattice points as

$$\alpha^{(j)} = \begin{bmatrix} \alpha_1^{(j)} \\ \alpha_2^{(j)} \end{bmatrix} = \frac{\mathrm{mod}\left((j-1)\begin{bmatrix} 1 \\ Z \end{bmatrix}, N\right)}{N}, \qquad j = 1, \ldots, N$$

Parametric models

Interpolatory model reduction:

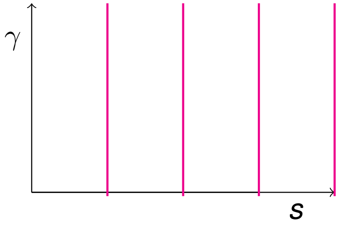
1. Discretize the parameter space
2. Build a reduced model for each node γ_j
3. Merge the spaces



For DPA

Parameter integrated in the algorithm:

- Vectors are functions of γ
- Eigenvalues and moments are functions of γ
- Compute these functions



For moment matching

Hermite interpolation

- For moment matching:
  - $\dfrac{\partial^i \widehat{H}(\sigma_j)}{\partial s^i} = \dfrac{\partial^i H(\sigma_j)}{\partial s^i}$ for $i = 0, \ldots, 2k-2$
  - $\dfrac{\partial^{i+1} \widehat{H}(\sigma_j)}{\partial \gamma \partial s^i} = \dfrac{\partial^{i+1} H(\sigma_j)}{\partial \gamma \partial s^i}$ for $i = 0, \ldots, k-1$

  [Baur, Beattie, Benner, Gugercin 2011] [Yue, M. 2012]
- For dominant poles:
  - $\lambda_j = \widehat{\lambda}_j$
  - $\dfrac{\partial \lambda_j}{\partial \gamma} = \dfrac{\partial \widehat{\lambda}_j}{\partial \gamma}$
  - $\widehat{H}(s) \sim H(s), s \to \lambda$
  - $\dfrac{\partial \widehat{H}(s)}{\partial \gamma} \sim \dfrac{\partial H(s)}{\partial \gamma}, s \to \lambda$

  [Saadvandi, M. & Desmet 2013]

Interpolatory MOR

- Standard algorithm:

  for $j = 1, \ldots, N$ **do**

  Compute $V_k^{(j)}$ and $W_k^{(j)}$ for node $\gamma^{(j)}$

  Merge the subspaces in V, W.

  end for

- Continuation DPA (CDPA)

  Compute V_k and W_k for node $\gamma^{(1)}$

  for $j = 1, \ldots, N$ **do**

  Use V ane W as a starting subspace (good starting values).

  Compute V_k and W_k for node $\gamma^{(j)}$.

  Merge the subspaces in V, W.

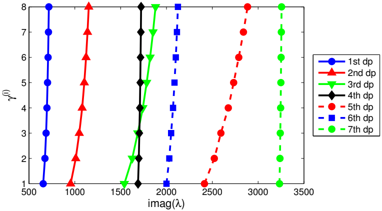
  end for

Numerical example: aluminum plate

Model:

$$(K_0 + \gamma_2 K_{\mathrm{bnd}} + i\omega(C_0 + \gamma_1 C_{\mathrm{bnd}}) - \omega^2 M)x = f$$

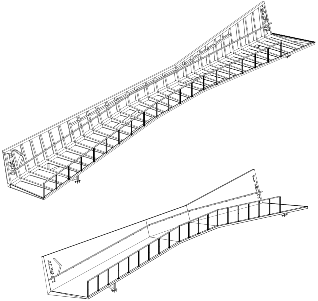
$$\begin{aligned} \gamma_1 &= 10^{-2} + \alpha_1(10^{-2} - 10^{-4}) \\ \gamma_2 &= \frac{-100}{\alpha_2 - 1} - 101 \\ \alpha_{1,2} &\in [0, 1) \end{aligned}$$

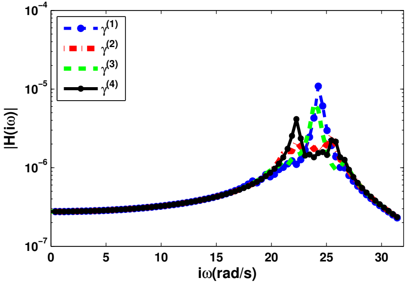


	Iterations
$\gamma^{(1)}$	22
$\gamma^{(2)}$	18
$\gamma^{(3)}$	13
$\gamma^{(4)}$	15
$\gamma^{(5)}$	10

Numerical example: Lamot footbridge

$$\left\{ \begin{array}{rcl} \left((1+0.02i)K_0 + (k_1 + i\omega c_1)K_1 - \omega^2 M\right) x & = & \\ y & = & \end{array} \right.$$





	# iterations	time (min)
$\gamma^{(1)}$	8	0.45
$\gamma^{(2)}$	4	0.25
$\gamma^{(3)}$	7	0.43
$\gamma^{(4)}$	3	0.19
Total	22	1.32

Tensorize the solution

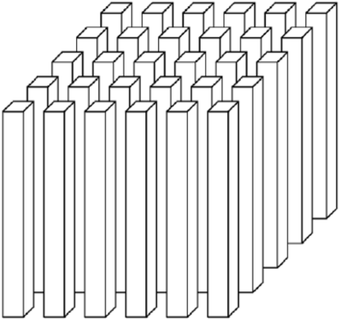
- Discretize the parameters in a grid of quadrature nodes:

$$\gamma^{(i_1,\ldots,i_p)} = (\gamma_1^{(i_1)}, \ldots, \gamma_p^{(i_p)}) \in \Gamma \subset \mathbb{R}^p$$

  with $i_j = 1, \ldots, n_j$, $j = 1, \ldots, p$.

- Then represent the state vector as a tensor:

$$\mathcal{X}_{:,i_1,\ldots,i_p} = x(\gamma^{(i_1,\ldots,i_p)})$$



- System for each grid point:

$$\left((A_0 + sB_0 + \sum_{j=1}^{p} \gamma_j^{(i_1,\ldots,i_p)} (A_j + sB_j)) \right) \mathcal{X}_{:,i_1,\ldots,i_p} = f$$

Tensor-Krylov

- For each grid point, a linear system needs to be solved:

$$\left(A_0 + sB_0 + \sum_{j=1}^{p} \gamma_j^{(i_1,\ldots,i_p)}(A_j + sB_j)\right) \mathcal{X}_{:,i_1,\ldots,i_p} \quad = \quad f$$

  This can be done with a tensor Krylov method [Kressner, Tobler 2011]: $\mathcal{X}$ is Tucker tensor

- Solve preconditioned system iteratively

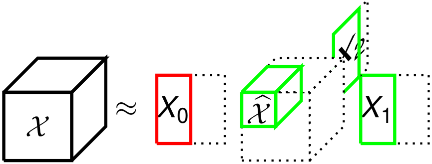
$$M^{-1}\left(A_0 + sB_0 + \sum_{j=1}^{p} \gamma_j^{(i_1,\ldots,i_p)}(A_j + sB_j)\right) x(\gamma^{(i_1,\ldots,i_p)}) = M^{-1} f$$

  The main operation is the matrix-vector product
  $t = M^{-1}(A(\gamma) + sB(\gamma))z$

Implementing preconditioned matrix-vector product

- Represent tensors using Tucker decomposition

$$X = X_1 \widehat{X} X_2^T \qquad \text{(SVD)}$$
$$\mathcal{X} = (X_0, X_1, \ldots, X_p) \bullet \widehat{\mathcal{X}}$$



- The operation $t = M^{-1}(A(\gamma) + sB(\gamma))z$ goes as follows
  1. No need to evaluate $M^{-1}(A(\gamma) + sB(\gamma))z$ for every grid point.
  2. z is represented as a low rank tensor $\mathcal{Z} \approx (Z_0, \ldots, Z_p) \bullet \widehat{\mathcal{Z}}$ with $Z_0 \in \mathbb{C}^{n \times r}$ and $r \ll n$.
  3. Sparse matvec operations: $M^{-1}(A_j + sB_j)Z_0, j = 0, \ldots, p$
  4. Operations on tensors: can be expensive!
- Key point: low rank X_0

Two algorithms

1. Interpolatory MOR
   - Tensor-Krylov with preconditioner M^{-1} with $M \approx A_0 + \sigma B_0$
   - Perform sufficiently large number of iterations so that parametric system is solved accurately enough
2. Small number of iterations:
   - Connection with multivariate Padé

Multivariate Padé

We want to build the parametric Krylov space:

$$(A(\gamma)+\sigma_0 B(\gamma))^{-1}f, (A(\gamma)+\sigma_1 B(\gamma))^{-1}f, \ldots, (A(\gamma)+\sigma_k B(\gamma))^{-1}f$$

where the basis vectors are discretized by tensors.

Theorem

Consider the power sequence ($M = A_0 + \sigma_j B_0$):

$$\begin{aligned} v_j^{(0)}(\gamma) &\equiv M^{-1}f \\ v_j^{(\ell)}(\gamma) &= v_j^{(\ell-1)}(\gamma) + \sum_{j=1}^{p} M^{-1}\gamma_j A_j v_j^{(\ell-1)}(\gamma) \end{aligned}$$

$v_j^{(\ell)}$ *is a moment matching polynomial of degree ℓ in γ:*

$$\|v_j^{(\ell)}(\gamma) - (A(\gamma)+\sigma_j B(\gamma))^{-1}f\|_2 = O(\|\gamma^{\ell+1}\|)$$

Reduced model

- Form a reduced model for the mean problem (using the tensors as state vectors.)

$$\left((A_0 + \sum_{j=1}^{p} \gamma_j^{(i_1,\ldots,i_p)} A_j) + sB_0\right) \mathcal{X}_{:,i_1,\ldots,i_p} = f$$

$$z = \sum_{i_1,\ldots,i_p} w_{i_1,\ldots,i_p}(c^* \mathcal{X}_{:,i_1,\ldots,i_p})$$

- From tensor representation (evaluation for grid points is available)

$$\mathcal{Y} = (c^*)\bullet_0 \mathcal{X}$$

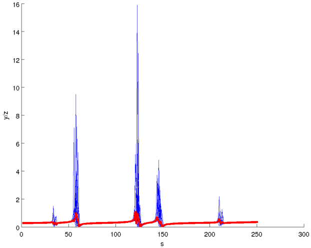
- Classical parametric model using the mode-0 vector space of iteration vectors: columns of $V_{j,0}$ in

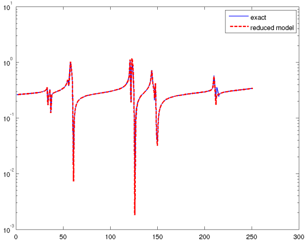
$$\mathcal{V}_j = (V_{j,0}, \ldots, V_{j,p})\bullet_0 \widehat{\mathcal{V}}_j$$

  span the moments.
  Project on the joined $V = \mathrm{orth}([V_{0,0}, \ldots, V_{k,0}])$

Numerical example: acoustic box

3D box with parametric Robin boundary condition on two faces





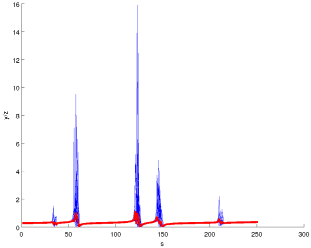
$\ell = 3$ GMRES iterations

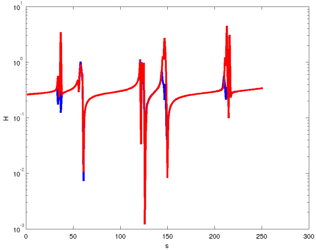
Tensors truncated $\tau = 10^{-8}$

Iter.	Rank	Iter.	Rank	Iter.	Rank
1	16	8	109	14	205
2	27	9	125	15	221
3	38	10	141	16	237
4	51	11	157	17	253
5	64	12	173	18	269
6	79	13	189	19	285
7	94				

Numerical example: acoustic box

3D box with parametric Robin boundary condition on two faces





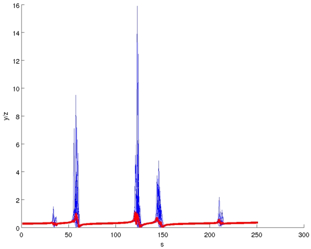
$\ell = 3$ GMRES iterations

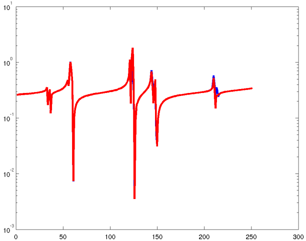
Tensors truncated $\tau = 10^{-4}$

Iter.	Rank	Iter.	Rank	Iter.	Rank
1	7	8	51	14	111
2	11	9	60	15	122
3	16	10	69	16	137
4	22	11	79	17	148
5	28	12	88	18	161
6	35	13	99	19	177
7	43				

Numerical example: acoustic box

3D box with parametric Robin boundary condition on two faces





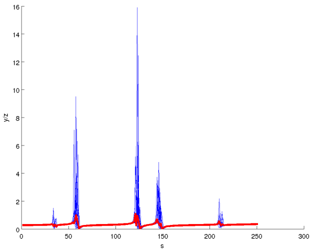
$\ell = 15$ GMRES iterations

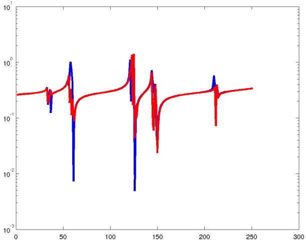
Tensors truncated $\tau = 10^{-4}$

Iter.	Rank	Iter.	Rank	Iter.	Rank
1	17	8	108	14	204
2	25	9	124	15	220
3	34	10	140	16	236
4	50	11	156	17	252
5	63	12	172	18	268
6	77	13	188	19	284
7	93				

Numerical example: acoustic box

3D box with parametric Robin boundary condition on two faces





M computed by ILU
$\ell = 15$ GMRES iterations
Tensors truncated $\tau = 10^{-4}$

Iter.	Rank	Iter.	Rank	Iter.	Rank
1	17	8	111	14	207
2	28	9	127	15	223
3	39	10	143	16	239
4	53	11	159	17	255
5	66	12	175	18	271
6	80	13	191	19	287
7	96				

Numerical example: footbridge

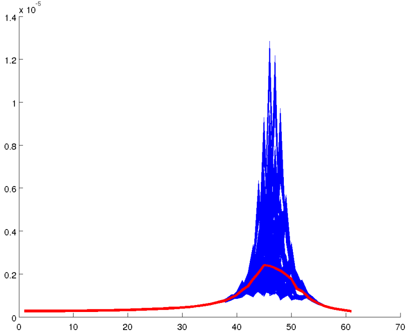
- The matrix dependency on the 8 parameters has rank 8

$$\left((1+0.02i)K_0+\sum_{i=1}^{4}(k_i+i\omega c_i)K_i-\omega^2 M\right)x=f$$

(Matrices $K_1,\ldots,K_4$ each have rank two.)

Ranks of iteration matrices in Arnoldi's method:

$n_i = 3$	$n_i = 5$
9	9
14	14
20	25
28	37
35	51



Results related to block Arnoldi for low rank parametric terms [Yue, M. 2013]

Outline

Conclusions

- Nonlinear in the frequency:
  - Rational Krylov with simple poles is easy
  - Rational Krylov with high order moment matching: possible, but technical (based on polynomial interpolation)
- Parametric:
  - Interpolatory MOR for DPA can enjoy continuation properties of eigenspaces
- Tensors:
  - Related to multivariate interpolation
  - If a low rank tensor approximation does not exist, it is not a practical method
  - Automatic rank reduction