On stability and passivity in model reduction: A generalized SVD-Krylov approach

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TECHNISCHE UNIVERSITÄT DRESDEN

Outline

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Our setting

Descriptor system $S := (E, A, B, C, D)$ with $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

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Standard system $(I, E^{-1}A, E^{-1}B, C, D)$ if E is regular

Model Reduction

Choose $k \ll n$, $W^\top$, $V \in \mathbb{R}^{n\times k}$
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In general, this do not hold for Krylov based methods

Our Idea

Set $W := V^\top X^\top$ where X solves a certain Lyapunov or Riccati equation.
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Generalization of

- SVD-Krylov based method,
- Spectral zero interpolation method.

Some definitions

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For regular E: admissible=stable

S and (E, A, C) are called

- **R-observable** if $\forall\lambda \in \mathbb{C}\colon\ \operatorname{rank}\left(\begin{smallmatrix} \lambda E - A \\ C \end{smallmatrix}\right) = n$,
- **I-observable** if $\operatorname{rank}\begin{pmatrix} E & A \\ 0 & E \\ 0 & C \end{pmatrix} = n + \operatorname{rank}(E)$.

Outline

Considered Lyapunov equation:

Reduced system $S_k = W \cdot S \cdot V = (WEV, WAV, WB, CV, D)$
with $W = V^\top X^\top$

Lemma 2.1 (Takaba et al. 1995)

Let S be admissible. Then there exists $X \in \mathbb{R}^{n \times n}$ such that

$$X^\top A + A^\top X + C^\top C = 0\,, \quad E^\top X \geq 0\,. \tag{1}$$

If E is regular then X is unique and $X = \mathfrak{O}_S E$.

Theorem 2.2 (K. 2013)

Let $X \in \mathbb{R}^{n\times n}$ *solve* (1) *and* $V \in \mathbb{R}^{n\times k}$. *We set* $S_k := V^\top X^\top \cdot S \cdot V$. *Then the following two are equivalent.*

(i) S_k *is stable,* (ii) S_k *is R-observable.*

If (E_k, A_k) *is regular then the following two are equivalent.*

(i') S_k *is impulse-free,* (ii') S_k *is I-observable.*

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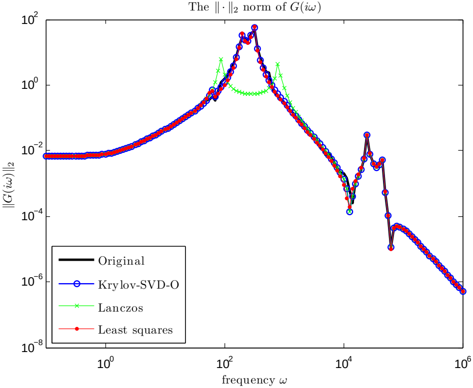
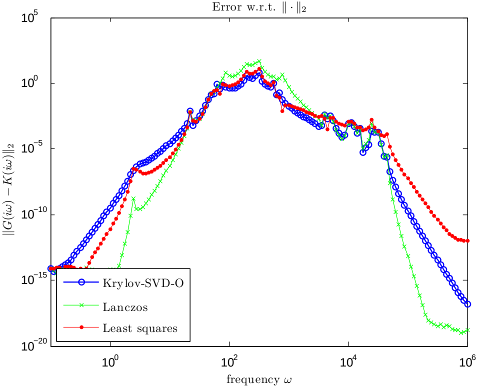
Proved in [Villemagne 1987] for stable and observable S with $E = I$

CD-Player, $n = 120$, $k = 12$, $s_0 = 0$, $s_1 = \infty$





Lanczos: 3 unstable poles

What if S_k in our approach is unstable or not impulse-free?

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Theorem 2.3 (K. 2013)

*Let S_k be unstable and (E_k, A_k) be regular. Then there exist $U, W \in \mathbb{R}^{k\times k}$ such that (**gQR**)*

$$U \cdot S_k \cdot W = \left(\begin{pmatrix} E_1 & E_2 \\ 0 & E_3 \end{pmatrix}, \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, (C_1, C_2), D \right)$$

where $\sigma(E_1, A_1) \subset \mathrm{i}\mathbb{R}$ and $\sigma(E_3, A_3) \subset \mathbb{C}_{<0} \cup \{\infty\}$.
Then $\widetilde{S}_k := (E_3, A_3, B_2, C_2, D)$ is a **stable** *and* **R-observable** *realization of G_k.*
Moreover, every I-observable realization of G_k is **impulse-free**.

Outline

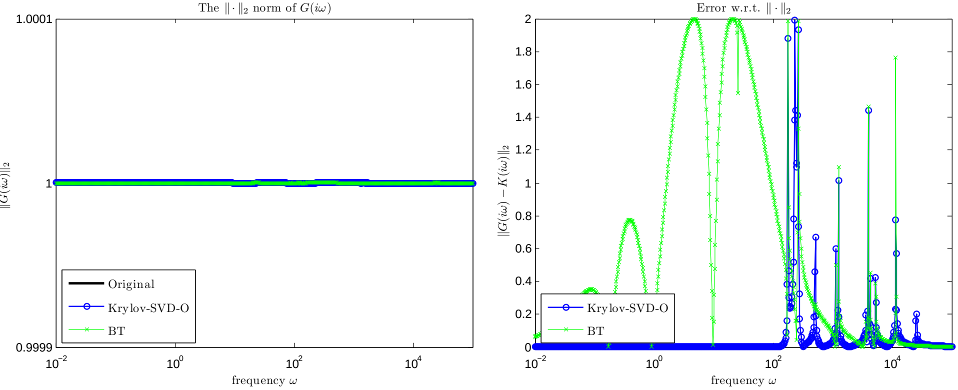
Definition

Let $m = p$ and $\sigma \in \mathbb{R}_{>0}$. Then G is called $\sigma-$**all-pass** if $G(\mathrm{i}\omega)^* G(\mathrm{i}\omega) = \sigma^2 I$ for $\omega \in \mathbb{R}$.

Theorem 3.1 (K. 2013)

Let S be controllable and $\sigma(E, -A) \cap \sigma(E, A) = \emptyset$. Let $\sigma \in \mathbb{R}_{>0}$ and G be $\sigma-$all-pass.
Set $S_k := V^\top E^\top \mathfrak{O}_S \cdot S \cdot V$ and let E_k be regular. Then G_k is $\sigma-$all-pass.
If $\sigma(E_k, -A_k) \cap \sigma(E_k, A_k) = \emptyset$ then S_k is minimal.

Random all-pass system, $n = 30$, $k = 10$, $s_0 = 10\mathrm{i}$



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Considered Riccati equation

Reduced system $S_k = W \cdot S \cdot V = (WEV, WAV, WB, CV, D)$
with $W = V^\top X^\top$

Lemma 4.1

Let (E, A) be regular and $D + D^\top > 0$. If there exists $X \in \mathbb{R}^{n\times n}$ such that

$$\begin{aligned} X^\top A + A^\top X + (C^\top - X^\top B)(D + D^\top)^{-1}(C - B^\top X) &= 0, \\ E^\top X &\geq 0. \end{aligned} \tag{2}$$

then S is passive.

Theorem 4.2 (Knockaert 2011)

Let $E = I$ *and* $X \in \mathbb{R}^{n \times n}$ *solve* (2) *with* $X > 0$. *Let* $V \in \mathbb{R}^{n \times k}$ *be injective. We set* $S_k := V^\top X \cdot S \cdot V$. *Then*

S_k *is passive.*

Theorem 4.2 (K. 2013)

Let $X \in \mathbb{R}^{n\times n}$ *solve* (2) *. Let* $V \in \mathbb{R}^{n\times k}$ *be injective. We set* $S_k := V^\top X^\top \cdot S \cdot V$*. If* (E_k, A_k) *is regular then* S_k *is passive.*

There is a similar result where ESPR and admissibility are always preserved.

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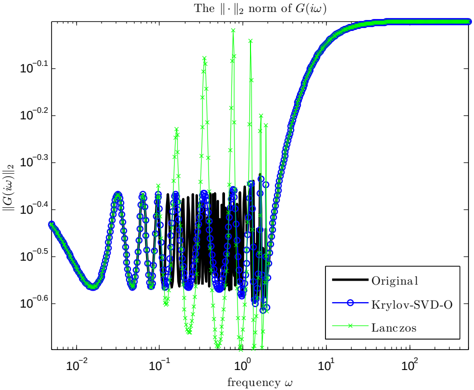
Remark

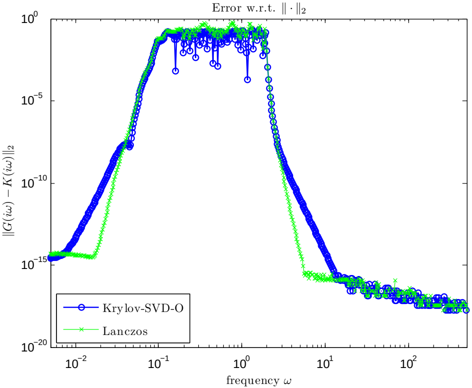
Now let

- X be an admissible solution of Riccati equation (2), i.e. $(E, A - B(D + D^\top)^{-1}(C - B^\top X))$ is admissible, and
- V span a **Krylov** space w.r.t. k spectral zeros of S in $\mathbb{C}_{<0}$.

Then S_k in Theorem 4.2 is the same as in the **spectral zero interpolation method** presented in [Antoulas 2005] and [Sorensen 2005].

RLC example in [Sorensen 2005], $n = 201$, $k = 20$, $s_0 = 0$, $s_1 = \infty$





Outlook

- How to avoid or deal with singular (E_k, A_k)?
- Generalize to systems with index > 1
- Preservation of minimum phase stability

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Thank you for your attention!

Selected References

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