Lehrstuhl für Regelungstechnik

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Cumulative model order reduction and solution of **Lyapunov equations** using **Krylov subspaces** and **adaptive shift selection**

MODRED

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Introduction

Linear, time invariant (LTI) system:

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \qquad \mathbf{E}, \mathbf{A} \in \mathbb{R}^{n\times n}, \ \mathbf{B} \in \mathbb{R}^{n\times m}$$

Goal: approximate the Laplace transform:

$$\mathbf{X}(s) := (s\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{B}$$

Model order reduction:

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \qquad \mathbf{C} \in \mathbb{R}^{p\times n}$$

$$\mathbf{G}(s) := \mathbf{C}\,(s\mathbf{E} - \mathbf{A})^{-1}\,\mathbf{B}$$

$r \ll n$

Lyapunov equation:

$$\mathbf{A}\mathbf{P}\mathbf{E}^T + \mathbf{E}\mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0}$$

$$\mathbf{P} = \mathbf{P}^T > \mathbf{0} \in \mathbb{R}^{n\times n}$$

Approximation:

$$\mathbf{X}(s) \approx \widehat{\mathbf{X}}(s) := \mathbf{V}\mathbf{X}_r(s)$$

$\approx$ $:=$

?

$$\mathbf{V} \in \mathbb{R}^{n\times r}$$

Reduced system:

$$\mathbf{X}_r(s) := (s\mathbf{E}_r - \mathbf{A}_r)^{-1}\,\mathbf{B}_r$$

$$\mathbf{E}_r, \mathbf{A}_r \in \mathbb{R}^{r\times r}, \ \mathbf{B}_r \in \mathbb{R}^{r\times m}$$

Krylov subspaces

Block Krylov subspace:

Shift/expansion point:

$s_0 \in \mathbb{C}$

Krylov block:

$\mathbf{v}_b(s_0) = (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{B} \quad \in \mathbb{C}^{n\times m}$

Krylov subspace:

$\mathcal{S} = \{s_1, \ldots, s_k\}$

$\mathcal{K}_b = \{\mathbf{v}_b(s_1), \ldots, \mathbf{v}_b(s_k)\}$

Basis:

$\mathrm{span}(\mathbf{V}) = \mathcal{K}_b$

$\mathbf{V} \in \mathbb{R}^{n\times r}, \quad r = km$

Tangential Krylov subspace:

Shift/expansion point:

$s_0 \in \mathbb{C}, \quad \mathbf{b}_0 \in \mathbb{C}^m$

Tangential Krylov direction:

$\mathbf{v}_t(s_0) = (\mathbf{A} - s_0\mathbf{E})^{-1}\mathbf{B}\mathbf{b}_0 \quad \in \mathbb{C}^{n}$

Krylov subspace:

$\mathcal{S} = \{s_1, \ldots, s_k\}, \ \mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$

$\mathcal{K}_t = \{\mathbf{v}_t(s_1), \ldots, \mathbf{v}_t(s_k)\}$

Basis:

$\mathrm{span}(\mathbf{V}) = \mathcal{K}_t$

$\mathbf{V} \in \mathbb{R}^{n\times r}, \quad r = k$

Projection: $\mathbf{W} \in \mathbb{R}^{n\times r}$ arbitrary

$$\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}, \quad \mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}, \quad \mathbf{B}_r = \mathbf{W}^T\mathbf{B}$$

Approximation:

$$\mathbf{X}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \approx \mathbf{V}\,(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r$$

Problem setting

Approximation:

$$\mathbf{X}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \approx \mathbf{V}\left(s\mathbf{E}_r - \mathbf{A}_r\right)^{-1}\mathbf{B}_r$$

Goal: kind of „salami slicing“ or „divide and conquer“

Error analysis

Approximation:

$$\mathbf{X}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} \approx \mathbf{V}\left(s\mathbf{E}_r - \mathbf{A}_r\right)^{-1}\mathbf{B}_r$$

Error:

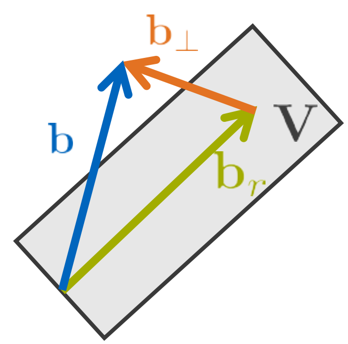
$$\mathbf{E}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B} - \mathbf{V}\left(s\mathbf{E}_r - \mathbf{A}_r\right)^{-1}\mathbf{B}_r$$

(Sum)

Krylov ↔ Sylvester:

$$\mathbf{AV} - \mathbf{EVS} = \mathbf{BL}, \quad \Lambda(\mathbf{S}) = \mathcal{S}$$

(Arnoldi algorithm)



[Gallivan, Vandendorpe, Van Dooren: *Sylvester equations and projection based model reduction*. Journal of Computational and Applied Mathematics, 162(1): 213-229, 2004]

$$\mathbf{AV} - \mathbf{EVE}_r^{-1}\mathbf{A}_r = \mathbf{B}_\perp\mathbf{L}, \quad \mathbf{B}_\perp = \mathbf{B} - \mathbf{EVE}_r^{-1}\mathbf{B}_r$$

[Wolf, Panzer, Lohmann: *Sylvester equations and a factorization of the error system in Krylov-based model reduction*. (MATHMOD), Vienna, Austria, 2012]

Error:

$$\mathbf{E}(s) = (s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}_\perp\left[\mathbf{L}\left(s\mathbf{E}_r - \mathbf{A}_r\right)^{-1}\mathbf{B}_r + \mathbf{I}\right]$$

(Factorization)

Cumulative approximation

Approximation:

$$
\begin{aligned}
\mathbf{X}(s) &= \mathbf{V}\mathbf{X}_r(s) + \mathbf{E}(s) \\
&= \mathbf{V}\,\underbrace{(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r} + \underbrace{(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}_\perp}\left[\mathbf{L}\,\underbrace{(s\mathbf{E}_r - \mathbf{A}_r)^{-1}\mathbf{B}_r} + \mathbf{I}\right] \\
&= \mathbf{V}\qquad \mathbf{X}_r(s) \qquad + \qquad \mathbf{X}_\perp(s) \qquad [\mathbf{L} \qquad \mathbf{X}_r(s) \qquad + \mathbf{I}]
\end{aligned}
$$

⋮

Approximate $\mathbf{X}_\perp(s)$ by $\mathbf{V}_2, \mathbf{W}_2$:

$$
= \mathbf{V}_1\mathbf{X}_{r1}(s) + \mathbf{V}_2\mathbf{X}_{r2}(s) + \quad \mathbf{X}_{\perp 2} \quad [\mathbf{L}_2\mathbf{X}_{r2}(s) + \mathbf{I}]\,[\mathbf{L}_1\mathbf{X}_{r1}(s) + \mathbf{I}]
$$

Total approximation:

$$
= \mathbf{V}_{\mathrm{tot}} \quad \mathbf{X}_{\mathrm{tot}}(s) \quad + \quad \mathbf{X}_{\perp,\mathrm{tot}}(s) \quad [\mathbf{L}_{\mathrm{tot}} \quad \mathbf{X}_{\mathrm{tot}}(s) \quad + \mathbf{I}]
$$

$$
\mathbf{V}_{\mathrm{tot}} = [\mathbf{V}_1, \mathbf{V}_2], \qquad \mathbf{L}_{\mathrm{tot}} = [\mathbf{L}_1, \mathbf{L}_2], \qquad \mathbf{X}_{\mathrm{tot}} = \begin{bmatrix} \mathbf{X}_{r1} \\ \mathbf{X}_{r2} \end{bmatrix}
$$

Cumulative approach

Cumulative approximation:

$$\mathbf{X}(s) = \mathbf{V}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{X}_{\perp,\mathrm{tot}}(s)\left[\mathbf{L}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{I}\right]$$

$$\mathbf{V}_{\mathrm{tot}} = [\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_k]$$

$$\mathbf{L}_{\mathrm{tot}} = [\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_k]$$

$$\mathbf{X}_{\mathrm{tot}}(s) = (s\mathbf{E}_{\mathrm{tot}} - \mathbf{A}_{\mathrm{tot}})^{-1}\mathbf{B}_{\mathrm{tot}}$$

$$\mathbf{A}_{\mathrm{tot}} = \begin{bmatrix} \mathbf{A}_{r1} & & & \\ \mathbf{B}_{r2}\mathbf{L}_1 & \mathbf{A}_{r2} & & \\ \vdots & \ddots & \ddots & \\ \mathbf{B}_{rk}\mathbf{L}_1 & \ldots & \mathbf{B}_{rk}\mathbf{L}_{k-1} & \mathbf{A}_{rk} \end{bmatrix}, \mathbf{E}_{\mathrm{tot}} = \begin{bmatrix} \mathbf{E}_{r1} & & \\ & \ddots & \\ & & \mathbf{E}_{rk} \end{bmatrix}, \mathbf{B}_{\mathrm{tot}} = \begin{bmatrix} \mathbf{B}_{r1} \\ \vdots \\ \mathbf{B}_{rk} \end{bmatrix}$$

Conclusions:

Numerically efficient (main effort is computation of $\mathbf{V}_i$)

Individual (decoupled) reduction steps

Degrees of freedom:

Set of shifts $\mathcal{S}_i$ for each $\mathbf{V}_i$

Direction of projection $\mathbf{W}_i$ **?**

Direction of projection

Proposition:

For given set $\mathcal{S}$, choose $\mathbf{W}$ such that: $\Lambda(-\mathbf{E}_r^{-1}\mathbf{A}_r) = \mathcal{S}$

Motivation:

Necessary condition for $\mathcal{H}_2$ optimal model order reduction ($\mathcal{H}_2$ pseudo-optimality)
Equivalent to $[\mathbf{L}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{I}]$ all-pass:

$$\mathbf{X}(s) = \mathbf{V}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{X}_{\perp,\mathrm{tot}}(s)\left[\mathbf{L}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{I}\right]$$

Implementation: Pseudo-Optimal Reduction by Krylov (PORK)

```
% select s = [s1, s2, ...]
[V,S,L] = arnoldi(E,A,B,s);
```

```
% select also b = [b1, b2, ...]
[V,S,L] = arnoldi(E,A,B,s,b);
```

```
X = lyap(-S,L'*L);
Br = - X\L';
Ar = S * Br*L';
Er = eye(size(Ar));
```

T. Wolf, H. Panzer, B. Lohmann: *H2 Pseudo-Optimality in Model Order Reduction by Krylov Subspace Methods*. Proceedings of the European Control Conference (ECC), Zurich, 2013

Application 1: Model order reduction

Output:

Arbitrary $\mathbf{C} \in \mathbb{R}^{p \times n}$, then $\mathbf{C}_r = \mathbf{C}\mathbf{V}$

Cumulative model reduction:

$$\mathbf{G}(s) = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B} = \mathbf{G}_{\text{tot}}(s) + \mathbf{G}_{\perp,\text{tot}}(s)\,\mathbf{G}_{\mathbf{L},\text{tot}}(s)$$

$$\mathbf{G}_{\text{tot}} = \mathbf{C}_{\text{tot}}\left(s\mathbf{E}_{\text{tot}} - \mathbf{A}_{\text{tot}}\right)^{-1}\mathbf{B}_{\text{tot}}$$

$$\mathbf{G}_{\perp,\text{tot}} = \mathbf{C}\left(s\mathbf{E} - \mathbf{A}\right)^{-1}\mathbf{B}_{\perp,\text{tot}}$$

$$\mathbf{G}_{\mathbf{L},\text{tot}} = \left[\mathbf{L}_{\text{tot}}\left(s\mathbf{E}_{\text{tot}} - \mathbf{A}_{\text{tot}}\right)^{-1}\mathbf{B}_{\text{tot}} + \mathbf{I}\right]$$

Published: Stability-Preserving, Adaptive Rational Krylov (SPARK)

Cumulative model reduction, SISO, Reduced order 2 in each "slice"

Trust-region optimization for complex s_i (ready-to-run MATLAB code)

H. Panzer, S. Jaensch, T. Wolf, B. Lohmann: *A Greedy Rational Krylov Method for H2-Pseudooptimal Model Order Reduction with Preservation of Stability*. Proceedings of the American Control Conference (ACC), Washington DC, 2013

Unpublished:

MIMO, Optimized trust-region algorithm

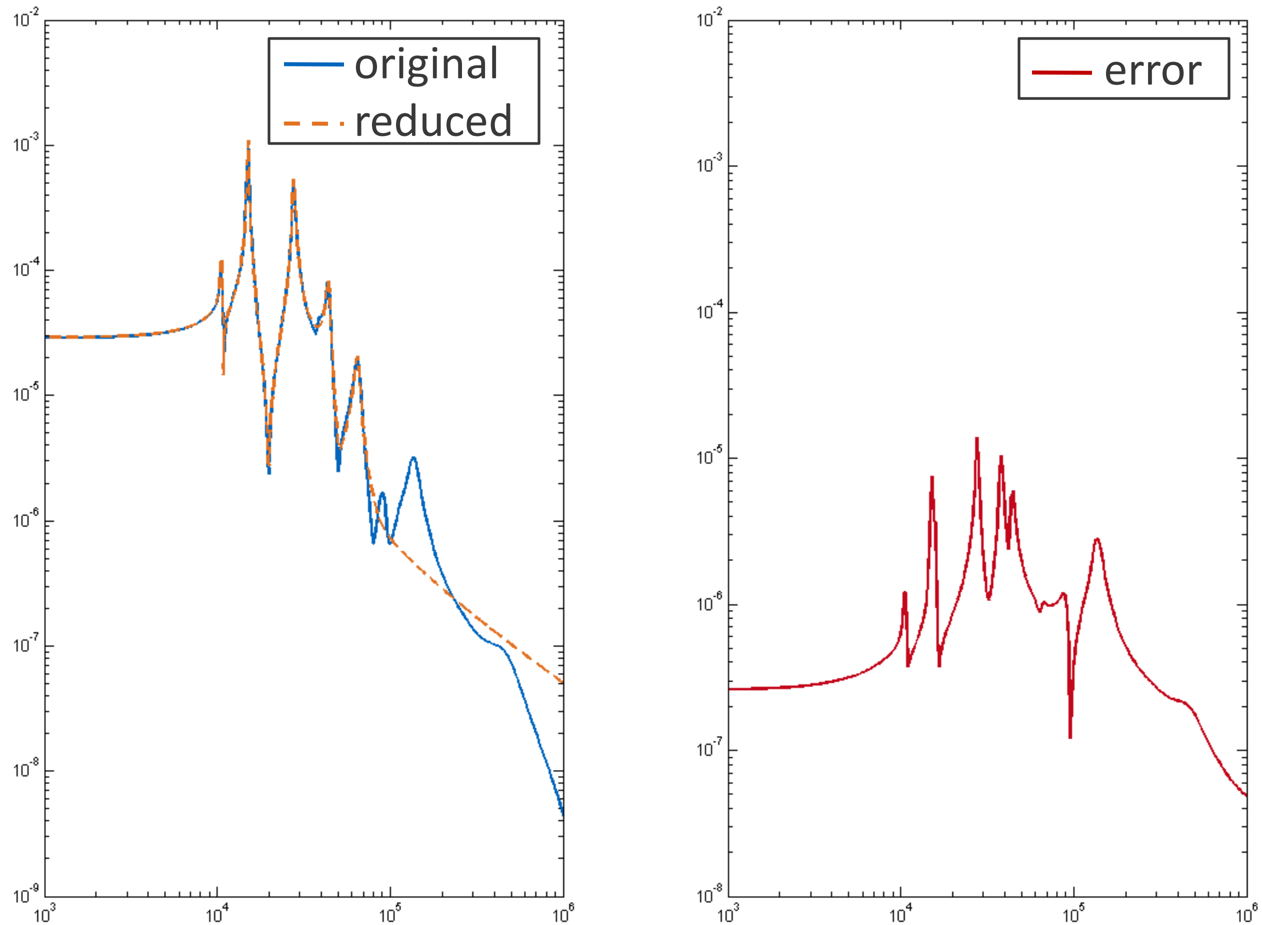
Application 1: Example

System:

„Butterfly gyro“
Input to first output
$n = 34\,722$

Oberwolfach model reduction benchmark collection, Oct.2003, Available online

Cumulative MOR:



$i = 7$

Application 2: Lyapunov equation

$$\mathbf{A}\mathbf{P}\mathbf{E}^T + \mathbf{E}\mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0}$$

Rational Krylov subspace method (RKSM):

$\mathbf{V}_{\text{tot}}$ // Basis

$\mathbf{A}_{\text{tot}},\ \mathbf{E}_{\text{tot}},\ \mathbf{B}_{\text{tot}}$ // PORK

$\mathbf{A}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{E}_{\text{tot}}^T + \mathbf{E}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{A}_{\text{tot}}^T + \mathbf{B}_{\text{tot}}\mathbf{B}_{\text{tot}}^T = \mathbf{0}$ // Direct solver

$\widehat{\mathbf{P}}_{\text{RKSM}} = \mathbf{V}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{V}_{\text{tot}}^T$ // Approximation

[V. Druskin, V. Simoncini: "Adaptive Rational Krylov Subspaces for Large-Scale Dynamical Systems." System & Control Letters Vol. 60, pp. 546-560 (2011)]

Alternating directions implicit (ADI) iteration:

$\mathbf{Z} := [\mathbf{Z}_1,\ \mathbf{Z}_2,\ \ldots,\ \mathbf{Z}_k]$ // ADI Basis

$\mathbf{Z}_1 = \sqrt{2\,\mathrm{Re}(s_1)}\,(\mathbf{A} - s_1\mathbf{E})^{-1}\,\mathbf{B}$ // Iteration

$$\mathbf{Z}_{i+1} = \sqrt{\frac{\mathrm{Re}(s_{i+1})}{\mathrm{Re}(s_i)}}\left(\mathbf{I} + (s_{i+1} + \overline{s}_i)(\mathbf{A} - s_{i+1}\mathbf{E})^{-1}\right)\mathbf{E}\mathbf{Z}_i, \qquad i = 2, \ldots, k$$

$\widehat{\mathbf{P}}_{\text{ADI}} = \mathbf{Z}\mathbf{Z}^H$ // Approximation

[J.-R. Li and J. White: "Low Rank Solution of Lyapunov Equations." SIAM. J. Matrix Anal. & Appl. Vol. 24, Issue 1, pp. 260-280 (2002)]

Application 2: Lyapunov equation

$$\mathbf{A}\mathbf{P}\mathbf{E}^T + \mathbf{E}\mathbf{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{0}$$

Rational Krylov subspace method (RKSM):

$$\mathbf{V}_{\text{tot}}$$

$$\mathbf{A}_{\text{tot}},\ \mathbf{E}_{\text{tot}},\ \mathbf{B}_{\text{tot}}$$

$$\mathbf{A}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{E}_{\text{tot}}^T + \mathbf{E}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{A}_{\text{tot}}^T + \mathbf{B}_{\text{tot}}\mathbf{B}_{\text{tot}}^T = \mathbf{0}$$

$$\widehat{\mathbf{P}}_{\text{RKSM}} = \mathbf{V}_{\text{tot}}\mathbf{P}_{\text{tot}}\mathbf{V}_{\text{tot}}^T$$

[V. Druskin, V. Simoncini: "Adaptive Rational Krylov Subspaces for Large-Scale Dynamical Systems." System & Control Letters Vol. 60, pp. 546-560 (2011)]

Alternating directions implicit (ADI) iteration:

$$\mathbf{Z} := [\mathbf{Z}_1,\ \mathbf{Z}_2,\ \ldots,\ \mathbf{Z}_k]$$

$$\mathbf{Z}_1 = \sqrt{2\,\mathrm{Re}(s_1)}\,(\mathbf{A} - s_1\mathbf{E})^{-1}\,\mathbf{B}$$

$$\mathbf{Z}_{i+1} = \sqrt{\frac{\mathrm{Re}(s_{i+1})}{\mathrm{Re}(s_i)}}\left(\mathbf{I} + (s_{i+1} + \overline{s}_i)(\mathbf{A} - s_{i+1}\mathbf{E})^{-1}\right)\mathbf{E}\mathbf{Z}_i$$

$$\widehat{\mathbf{P}}_{\text{ADI}} = \mathbf{Z}\mathbf{Z}^H$$

[J.-R. Li and J. White: "Low Rank Solution of Lyapunov Equations." SIAM. J. Matrix Anal. & Appl. Vol. 24, Issue 1, pp. 260-280 (2002)]

Link:

Theorem:

$$\widehat{\mathbf{P}}_{\text{RKSM}} = \widehat{\mathbf{P}}_{\text{ADI}}$$

if and only if

$$\Lambda(-\mathbf{E}_{\text{tot}}^{-1}\mathbf{A}_{\text{tot}}) = \mathcal{S}$$

Corollary:

- 2 ways for computing the same approximation
- virtual reduced system can be associated to ADI
- $\mathcal{H}_2$ pseudo-optimality

[G.M. Flagg, S. Gugercin : "On the ADI method for the Sylvester equation and the optimal-H2 points " Applied Numerical Mathematics, Elsevier, 2013, 64]

Re-formulation of ADI iteration

Residual in Lypunov equation:

$$\mathbf{R} := \mathbf{A}\widehat{\mathbf{P}}\mathbf{E}^T + \mathbf{E}\widehat{\mathbf{P}}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = \mathbf{B}_\perp \mathbf{B}_\perp^T$$

Original formulation:

$$\mathbf{Z}_1 = \sqrt{2\,\mathrm{Re}(s_1)}\,(\mathbf{A} - s_1\mathbf{E})^{-1}\,\mathbf{B}$$

$$\mathbf{Z}_{i+1} = \sqrt{\frac{\mathrm{Re}(s_{i+1})}{\mathrm{Re}(s_i)}}\left(\mathbf{I} + (s_{i+1} + \overline{s}_i)(\mathbf{A} - s_{i+1}\mathbf{E})^{-1}\right)\mathbf{E}\mathbf{Z}_i, \qquad i = 2, \ldots, k$$

Re-formulated ADI: $\mathbf{B}_{\perp,0} := \mathbf{B}$

$$\mathbf{Z}_i = \sqrt{2\,\mathrm{Re}(s_i)}\,(\mathbf{A} - s_i\mathbf{E})^{-1}\,\mathbf{B}_{\perp,i-1}$$

$$\mathbf{B}_{\perp,i} = \mathbf{B}_{\perp,i-1} + \sqrt{2\,\mathrm{Re}(s_i)}\,\mathbf{E}\mathbf{Z}_i \qquad i = 1, \ldots, k$$

Calculation of the residual on the fly, without additional numerical effort

[P. Benner, P. Kürschner: *Computing Real Low-rank Solutions of Sylvester equations by the Factored ADI Method*. Max Planck Institute Magdeburg Preprint MPIMD/13-05, 2013]

[T. Wolf, H. Panzer, B. Lohmann: *ADI iteration for Lyapunov equations: A tangential approach and adaptive shift selection*. Pre-print: arXiv: 1312.1142. Dec, 2013]

Re-formulation of ADI iteration

Original formulation:

$$\mathbf{Z}_1 = \sqrt{2\,\mathrm{Re}(s_1)}\,(\mathbf{A} - s_1\mathbf{E})^{-1}\,\mathbf{B}$$

$$\mathbf{Z}_{i+1} = \sqrt{\frac{\mathrm{Re}(s_{i+1})}{\mathrm{Re}(s_i)}}\left(\mathbf{I} + (s_{i+1} + \overline{s}_i)(\mathbf{A} - s_{i+1}\mathbf{E})^{-1}\right)\mathbf{E}\mathbf{Z}_i, \qquad i = 2, \ldots, k$$

Re-formulated ADI: $\mathrm{B}_{\perp,0} := \mathrm{B}$

$$\mathbf{Z}_i = \sqrt{2\,\mathrm{Re}(s_i)}\,(\mathbf{A} - s_i\mathbf{E})^{-1}\,\mathbf{B}_{\perp,i-1}\mathbf{b}_i, \quad \|\mathbf{b}_i\|_2 = 1$$

$$\mathbf{B}_{\perp,i} = \mathbf{B}_{\perp,i-1} + \sqrt{2\,\mathrm{Re}(s_i)}\,\mathbf{E}\mathbf{Z}_i\,\mathbf{b}_i^H \qquad i = 1, \ldots, k$$

[T. Wolf, H. Panzer, B. Lohmann: *ADI iteration for Lyapunov equations: A tangential approach and adaptive shift selection*. Pre-print: arXiv: 1312.1142. Dec, 2013]

Tangential directions:

$\mathbf{b}_i \in \mathbb{C}^m \quad \Rightarrow \quad \mathbf{Z}_i \to \mathbf{z}_i \in \mathbb{C}^n$

Generalization:

Same shift m times with different tangential directions: Block-ADI

Tangential ADI iteration

Implementation: Tangential low-rank ADI (T-LR-ADI) iteration

Algorithm 1 Tangential-Low-Rank-ADI (T-LR-ADI)

Input: $\mathbf{E}$, $\mathbf{A}$, $\mathbf{B}$, tol

Output: Approximation $\widehat{\mathbf{P}} = \mathbf{Z}\mathbf{Z}^T$ and residual $\mathbf{R} = \mathbf{B}_\perp \mathbf{B}_\perp^T$

initial choice of $s_1 \in \mathbb{C}$ and $\mathbf{b}_1 \in \mathbb{C}^m$ with $\|\mathbf{b}_1\|_2 = 1$
$\mathbf{Z} = [\,]$, $\mathbf{B}_\perp = \mathbf{B}$
repeat
    solve $(\mathbf{A} - s_i\mathbf{E})\,\mathbf{y} = \mathbf{B}_\perp \mathbf{b}_i$ for $\mathbf{y}$
    if $s_i \in \mathbb{R}$ **then**
        $\mathbf{Z}_i = \sqrt{2s_i}\mathbf{y},\ \ \mathbf{L}_i = \sqrt{2s_i}\mathbf{b}_i$
    else
        $\alpha = \mathbf{b}_i^H \overline{\mathbf{b}}_i \frac{\mathrm{Re}(s_i)}{\overline{s}_i}, \quad \beta = \frac{1}{\sqrt{1-\alpha\overline{\alpha}}}, \quad \gamma = \sqrt{1+\mathrm{Re}(\alpha)}$
        $\mathbf{Z}_i = \frac{2}{\gamma}\sqrt{\mathrm{Re}(s_i)}\left[\mathrm{Re}(\mathbf{y}),\ \beta\left(\mathrm{Im}(\alpha)\,\mathrm{Re}(\mathbf{y}) + \gamma^2\,\mathrm{Im}(\mathbf{y})\right)\right]$
        $\mathbf{L}_i = \frac{2}{\gamma}\sqrt{\mathrm{Re}(s_i)}\left[\mathrm{Re}(\mathbf{b}_i),\ \beta\left(\mathrm{Im}(\alpha)\,\mathrm{Re}(\mathbf{b}_i) + \gamma^2\,\mathrm{Im}(\mathbf{b}_i)\right)\right]^T$
    end if
    $\mathbf{Z} = [\mathbf{Z}, \mathbf{Z}_i]$
    $\mathbf{B}_\perp = \mathbf{B}_\perp + \mathbf{E}\mathbf{Z}_i\mathbf{L}_i^T$
    determine s_{i+1} and $\mathbf{b}_{i+1}$ with $\|\mathbf{b}_{i+1}\|_2 = 1$
until $\|\mathbf{R}\|_2 = \max \Lambda\left(\mathbf{B}_\perp^T \mathbf{B}_\perp\right) < tol\,\|\mathbf{B}^T\mathbf{B}\|_2$

[T. Wolf, H. Panzer, B. Lohmann: *ADI iteration for Lyapunov equations: A tangential approach and adaptive shift selection*. Pre-print: arXiv: 1312.1142. Dec, 2013]

Application 2: Example

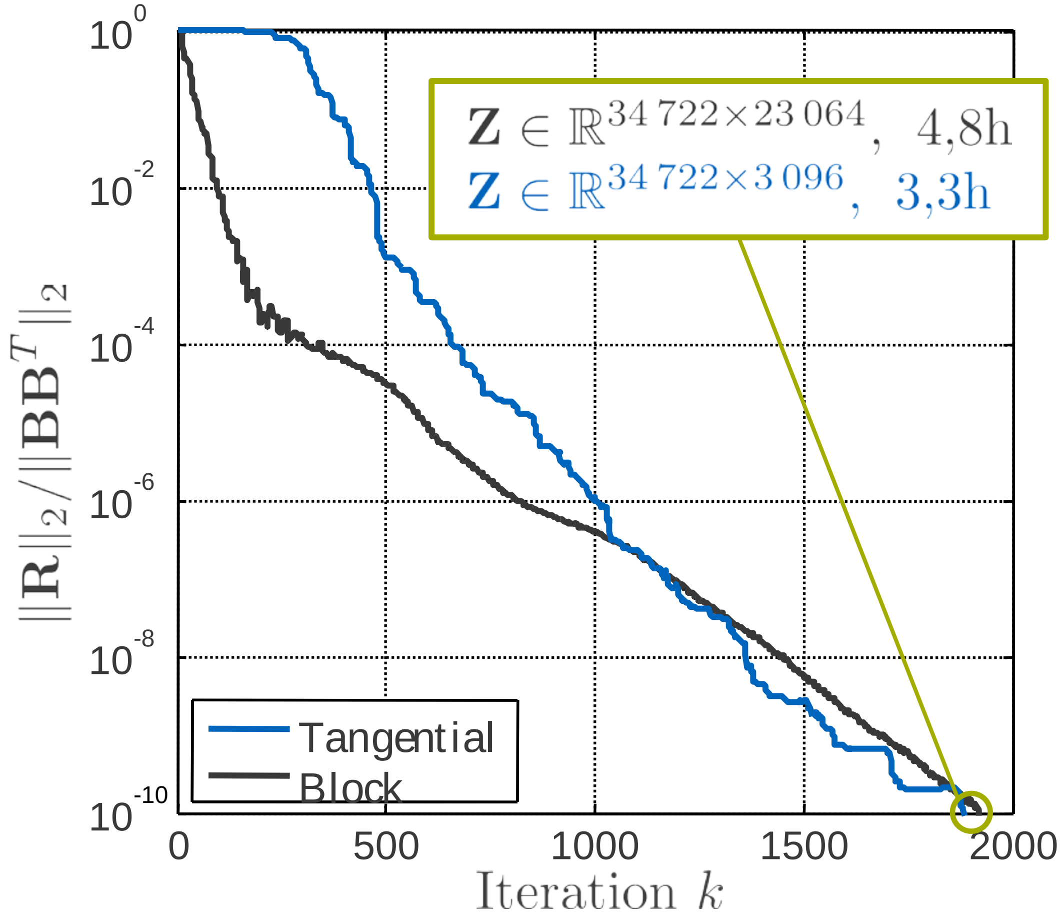
System:

„Butterfly gyro“
Observability Gramian

$$n = 34\,722$$
$$m = 12$$

Oberwolfach model reduction benchmark collection, Oct.2003, Available online

Comparison block ↔ tangential:



P. Benner, P. Kürschner, J. Saak: *Self-Generating and Efficient Shift Parameters in ADI Methods for Large Lyapunov and Sylvester Equations*. Max Planck Institute Magdeburg Preprint, October 2013

Conclusions

Cumulative approximation:

$$\mathbf{X}(s) = \mathbf{V}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{X}_{\perp,\mathrm{tot}}(s)\left[\mathbf{L}_{\mathrm{tot}}\mathbf{X}_{\mathrm{tot}}(s) + \mathbf{I}\right]$$

PORK:

$\mathcal{H}_2$ pseudo-optimal approximation

Model order reduction:

Adaptive choice of reduced order
Individually reduced systems
Trust-region algorithm for (adaptive) shift selection

Lyapunov equation:

Link between ADI and RKSM
Tangential ADI iteration
Adaptive shift selection

Application 2: Example

System:

„Steel Profile“
Controlability Gramian

$$n = 1357$$
$$m = 7$$

Oberwolfach model reduction benchmark collection, Oct.2003, Available online at [http://portal.uni-freiburg.de/imtek simulation/downloads/benchmark](http://portal.uni-freiburg.de/imtek)

Comparison block ↔ tangential: