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Projection-Free Balanced Truncation for Differential-Algebraic Systems

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2. Lyapunov balanced truncation
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Descriptor system

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t),\\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

Objective

Reduce to small size system

$$
\begin{aligned}
\widehat{E}\dot{\widehat{x}}(t) &= \widehat{A}\widehat{x}(t) + \widehat{B}u(t),\\
y(t) &= \widehat{C}\widehat{x}(t) + \widehat{D}u(t)
\end{aligned}
$$

under preservation of “special properties”

The “system space”

$\mathcal{V}, \widetilde{\mathcal{V}}$

$$\mathcal{B} = \left\{ \begin{pmatrix} x(\cdot) \\ u(\cdot) \end{pmatrix} \in C(\mathbb{R}, \mathbb{R}^{n+m}) \;:\; E\dot{x}(\cdot) = Ax(\cdot) + Bu(\cdot) \right\}. \qquad \text{(behavior)}$$

$$\mathcal{V} = \left\{ \begin{pmatrix} x(0) \\ u(0) \end{pmatrix} \;:\; \begin{pmatrix} x(\cdot) \\ u(\cdot) \end{pmatrix} \in \mathcal{B} \right\}, \qquad \widetilde{\mathcal{V}} = \left\{ x \;:\; \begin{pmatrix} x \\ u \end{pmatrix} \in \mathcal{V} \right\}. \qquad \text{(system spaces)}$$

System spaces of the dual system $E^T\dot{x} = A^Tx + C^Tu$ is denoted by $\mathcal{V}^*$, $\widetilde{\mathcal{V}}^*$.

If the system is impulse controllable, then

$$\widetilde{\mathcal{V}} = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in \mathbb{R}^{n+m} \;:\; Ax + Bu \in \operatorname{im} E \right\}.$$

Notation

For $F, G \in \mathbb{R}^{k,k}$, and $\mathcal{W} \subset \mathbb{R}^k$, we write $F =_{\mathcal{W}} G$, if

$$x^TFy = x^TGy \quad \forall x, y \in \mathcal{W}.$$

Balancing-related model reduction

Given

$P, Q \in \mathbb{R}^{n,n}$ such that

$$EP^* =_{\widetilde{\mathcal{V}}^*} PE^* \geq 0, \qquad \operatorname{im} P \subset \widetilde{\mathcal{V}}^*,$$
$$E^*Q =_{\widetilde{\mathcal{V}}} Q^*E \geq 0, \qquad \operatorname{im} Q^* \subset \widetilde{\mathcal{V}}.$$

Balancing

Find regular W, T such that

$$WET = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad WP(T^*)^{-1} = (W^*)^{-1}QT = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix},$$

where $\Sigma \geq 0$ is diagonal.

Balanced truncation

Balancing and partitioning

Find regular W, T such that

$$WET = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad WAT = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{A}_{23} \\ \widetilde{A}_{31} & \widetilde{A}_{32} & \widetilde{A}_{33} \end{bmatrix},$$

$$WB = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{B}_3 \end{bmatrix}, \qquad CT = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 & \widetilde{C}_3 \end{bmatrix},$$

$$WP(T^*)^{-1} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (W^*)^{-1}QT = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Balanced truncation

Balancing and partitioning

Identify small singular values

$$WET = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad WAT = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} & \widetilde{A}_{13} \\ \widetilde{A}_{21} & \widetilde{A}_{22} & \widetilde{A}_{23} \\ \widetilde{A}_{31} & \widetilde{A}_{32} & \widetilde{A}_{33} \end{bmatrix},$$

$$WB = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_2 \\ \widetilde{B}_3 \end{bmatrix}, \qquad CT = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_2 & \widetilde{C}_3 \end{bmatrix},$$

$$WP(T^*)^{-1} = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad (W^*)^{-1}QT = \begin{bmatrix} \Sigma_1 & 0 & 0 \\ 0 & \Sigma_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Balanced truncation

Truncation

$$\widehat{E} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widehat{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{13} \\ \widetilde{A}_{31} & \widetilde{A}_{33} \end{bmatrix},$$

$$\widehat{B} = \begin{bmatrix} \widetilde{B}_1 \\ \widetilde{B}_3 \end{bmatrix}, \qquad \widehat{C} = \begin{bmatrix} \widetilde{C}_1 & \widetilde{C}_3 \end{bmatrix},$$

$$\widehat{P} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \widehat{Q} = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Reduced System

$$\widehat{E}\dot{\widehat{x}}(t) = \widehat{A}\widehat{x}(t) + \widehat{B}u(t),$$
$$y(t) = \widehat{C}\widehat{x}(t) + Du(t)$$

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Lyapunov balanced truncation I

Assumptions

System is of index one and asymptotically stable.

Lyapunov balanced descriptor system

Equations to be balanced:

$$AP^* + PA^* + BB^* =_{\widetilde{\mathcal{V}}^*} 0, \quad EP^* =_{\widetilde{\mathcal{V}}^*} PE^*$$
$$A^*Q + Q^*A + C^*C =_{\widetilde{\mathcal{V}}} 0, \quad E^*Q =_{\widetilde{\mathcal{V}}} Q^*E$$

Bound

The reduced system fulfills

$$\|G - \widehat{G}\|_{\mathcal{H}^\infty} \leq 2\,\mathrm{trace}\,\Sigma_2.$$

Lyapunov balanced truncation II

$$E = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \quad D = D \quad \widehat{=} \quad \begin{matrix} A_{11} - A_{12}A_{22}^{-1}A_{21}, \ B_1 - A_{12}A_{22}^{-1}B_2, \\ C_1 - C_2A_{22}^{-1}A_{21}, \ D - C_2A_{22}^{-1}B_2 \end{matrix}$$

Sketch of the proof

$$\begin{array}{ccc} \text{descriptor system} & \xrightarrow[\text{for descriptor system}]{\text{Lyapunov truncation}} & \text{reduced descriptor system} \\ \| & & \| \\ \text{state space system} & \xrightarrow[\text{for state space system}]{\text{Lyapunov truncation}} & \text{reduced state space system} \end{array}$$

Remark

$$G(\infty) = \widehat{G}_r(\infty)$$

Singular perturbation I

Idea

- Lyapunov balanced truncation for frequency inverted system (transfer function $G(s^{-1})$)
- Matches the static gain of the system ($G(0) = \widehat{G}(0)$)
- Gramians of the frequency inverted system are the same
- Implies same error bound
- Reduced system (in the ODE case)

$$\widehat{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \qquad \widehat{B} = B_1 - A_{12}A_{22}^{-1}B_2,$$
$$\widehat{C} = C_1 - C_2A_{22}^{-1}A_{21}, \qquad \widehat{D} = D - C_2A_{22}^{-1}B_2.$$

Singular perturbation II

Reduced system (DAE case)

$A_S = \begin{bmatrix} \widetilde{A}_{22} & \widetilde{A}_{23} \\ \widetilde{A}_{32} & \widetilde{A}_{33} \end{bmatrix}$ and $\widehat{E} = I$:

$$\widehat{A} = \widetilde{A}_{11} - \begin{bmatrix} \widetilde{A}_{12} & \widetilde{A}_{13} \end{bmatrix} A_S^{-1} \begin{bmatrix} \widetilde{A}_{21} \\ \widetilde{A}_{31} \end{bmatrix}, \qquad \widehat{B} = \widetilde{B}_1 - \begin{bmatrix} \widetilde{A}_{12} & \widetilde{A}_{13} \end{bmatrix} A_S^{-1} \begin{bmatrix} \widetilde{B}_2 \\ \widetilde{B}_3 \end{bmatrix},$$

$$\widehat{C} = \widetilde{C}_1 - \begin{bmatrix} \widetilde{C}_2 & \widetilde{C}_3 \end{bmatrix} A_S^{-1} \begin{bmatrix} \widetilde{A}_{21} \\ \widetilde{A}_{31} \end{bmatrix}, \qquad \widehat{D} = D - \begin{bmatrix} \widetilde{C}_2 & \widetilde{C}_3 \end{bmatrix} A_S^{-1} \begin{bmatrix} \widetilde{B}_2 \\ \widetilde{B}_3 \end{bmatrix}.$$

Singular perturbation III

Sketch of the proof

$$\begin{array}{ccc} \text{descriptor system} & \xrightarrow[\text{for descriptor system}]{\text{singular perturbation truncation}} & \text{reduced state space system} \\ \Vert & & \Vert \\ \text{state space system} & \xrightarrow[\text{for state space system}]{\text{singular perturbation truncation}} & \text{reduced state space system} \end{array}$$

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Bounded real balanced truncation I

Assumptions

System is bounded real, that is $\|y\|_{L_2} \leq \|u\|_{L_2}$
$(\Leftrightarrow 0 \leq I - G(s)^* G(s)\ \forall s \in \mathbb{C}_+)$

Bounded real Lur'e equations

$$\begin{bmatrix} AP^* + PA^* + BB^* & PC^* + BD^* \\ CP^* + DB^* & DD^* - I \end{bmatrix} + \begin{bmatrix} K_B \\ L_B \end{bmatrix} \begin{bmatrix} K_B^* & L_B^* \end{bmatrix} =_{\mathcal{V}^*} 0,$$

$$EP^* =_{\mathcal{V}^*} PE^*$$

$$\begin{bmatrix} A^*Q + Q^*A + C^*C & Q^*B + C^*D \\ B^*Q + D^*C & D^*D - I \end{bmatrix} + \begin{bmatrix} K_C^* \\ L_C^* \end{bmatrix} \begin{bmatrix} K_C & L_C \end{bmatrix} =_{\mathcal{V}} 0,$$

$$E^*Q =_{\mathcal{V}} Q^*E$$

Bounded real balanced truncation II

Bound

With

$$H(s) = \begin{bmatrix} C \\ K_C \end{bmatrix} (sE - A)^{-1} \begin{bmatrix} B & K_B \end{bmatrix} = \begin{bmatrix} G(s) & * \\ * & * \end{bmatrix},$$

$$\widehat{H}(s) = \begin{bmatrix} \widehat{C} \\ \widehat{K_C} \end{bmatrix} (s\widehat{E} - \widehat{A})^{-1} \begin{bmatrix} \widehat{B} & \widehat{K_B} \end{bmatrix} = \begin{bmatrix} \widehat{G}(s) & * \\ * & * \end{bmatrix}$$

holds

$$\|G - \widehat{G}\|_{\mathcal{H}^\infty} \leq \|H - \widehat{H}\|_{\mathcal{H}^\infty} \leq 2\,\mathrm{trace}\,\Sigma_2.$$

Follows from the Lyapunov equations inside the Lur'e equations.

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Gap metric

Definition

Let $\mathcal{V}, \mathcal{W} \subset X$ be subspaces of the Hilbert space X.

$$\delta(\mathcal{V}, \mathcal{W}) = \max_{v \in \mathcal{V}, \|v\|=1} \min_{w \in \mathcal{W}} \|v - w\|.$$

$$\widehat{\delta}(\mathcal{V}, \mathcal{W}) = \max\{\delta(\mathcal{V}, \mathcal{W}), \delta(\mathcal{W}, \mathcal{V})\}$$

Gap metric between two systems:
Gap between the graph (u, y) of the systems.

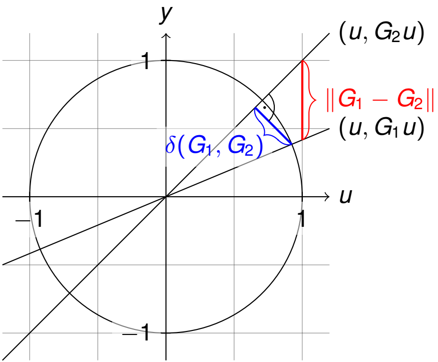
Remarks

- If $G_1, G_2 \in \mathcal{H}_\infty$, then

$$\widehat{\delta}(G_1, G_2) \leq \|G_1 - G_2\|_\infty$$

- Invariant to orthogonal input-output transformations

$$\begin{pmatrix} u \\ y \end{pmatrix} \to U \cdot \begin{pmatrix} u \\ y \end{pmatrix}.$$


y
$(u, G_2 u)$
1
$\|G_1 - G_2\|$
$(u, G_1 u)$
$\delta(G_1, G_2)$
u
-1
1
-1

Moebius transform

$$\begin{bmatrix} u \\ y \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1}{\sqrt{2}} \cdot (y+u) \\ \frac{1}{\sqrt{2}} \cdot (y-u) \end{bmatrix}.$$

The transfer function behaves like

$$G(s) \rightsquigarrow \mathcal{M}(G)(s) := (I - G(s))(I + G(s))^{-1}.$$

Properties

Orthogonal transformation on the graph of the system (preserves the gap).

$G(s)$ is bounded real $\Leftrightarrow$ $\mathcal{M}(G)$ is positive real.

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Positive real balanced truncation I

Assumptions

G is positive real ($G(s)^* + G(s) \geq 0 \quad \forall s \in \mathbb{C}_+$)

Positive real balanced descriptor system

Equations to be balanced with unknown K_B, L_B, K_C, L_C:

$$\begin{bmatrix} AP^* + PA & PC^* - B \\ CP^* - B^* & -D - D^* \end{bmatrix} + \begin{bmatrix} K_B \\ L_B \end{bmatrix} \begin{bmatrix} K_B^* & L_B^* \end{bmatrix} =_{\mathcal{V}^*} 0,$$

$$EP^* =_{\mathcal{V}^*} PE^*$$

$$\begin{bmatrix} A^*Q + Q^*A & Q^*B - C^* \\ B^*Q - C & -D^* - D \end{bmatrix} + \begin{bmatrix} K_C^* \\ L_C^* \end{bmatrix} \begin{bmatrix} K_C & L_C \end{bmatrix} =_{\mathcal{V}} 0,$$

$$E^*Q =_{\mathcal{V}} Q^*E$$

Positive real balanced truncation II

Bound

$$\delta(G, \widehat{G}) \leq 2\,\mathrm{trace}\,\Sigma_2.$$

Sketch of the proof

$$\begin{array}{ccc} G \text{ (positive real)} & \xrightarrow[\text{for positive real}]{\text{truncation}} & \widehat{G} \text{ (positive real)} \\ {\scriptstyle \widetilde{u}=\sqrt{2}(y+u) \atop \widetilde{y}=\sqrt{2}(y-u)} \big\downarrow & & \big\downarrow {\scriptstyle \widetilde{u}=\sqrt{2}(y+u) \atop \widetilde{y}=\sqrt{2}(y-u)} \\ G_{\mathrm{BR}} \text{ (bounded real)} & \xrightarrow[\text{for bounded real}]{\text{truncation}} & \widehat{G}_{\mathrm{BR}} \text{ (bounded real)} \end{array}$$

Same “Gramians” for both systems.

$$\delta(G, \widehat{G}) = \delta(G_{\mathrm{BR}}, \widehat{G}_{\mathrm{BR}}) \leq \|G_{\mathrm{BR}} - \widehat{G}_{\mathrm{BR}}\|_{\mathcal{H}^\infty} \leq 2\,\mathrm{trace}\,\Sigma_2$$

Conclusion

- balanced truncation for DAEs based on Lyapunov and Lur'e equations
- standard, bounded real, positive real, singular perturbation

Outlook

- numerical solution of descriptor Lur'e equations
- frequency-weighted balanced truncation
- stochastic balanced truncation