Multiobjective PDE-constrained optimization using the reduced basis method

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TECHNISCHE UNIVERSITÄT DARMSTADT

SEVENTH FRAMEWORK PROGRAMME

Outline

- Short motivation on model order reduction
- Brief overview on the Reduced basis method
- Multiobjective Optimization
- Numerical results

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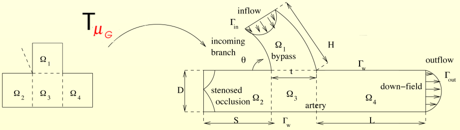
- Short motivation on model order reduction
- Brief overview on the Reduced basis method
- Multiobjective Optimization
  - ⋆ Pareto optimal solutions computed with the Reduced Basis method
  - ⋆ Sensitivity analysis for an efficient solution of the problem
- Numerical results

Motivation

$$\begin{cases} -\nu\Delta\mathbf{u}+\nabla p=0 & \text{in } \Omega \\ \nabla\cdot\mathbf{u}=0 & \text{in } \Omega \\ \mathbf{u}=0 & \text{on } \Gamma_D \\ \nu\dfrac{\partial\mathbf{u}}{\partial\mathbf{n}}-p\mathbf{n}=-\mathbf{n} & \text{on } \Gamma_{in} \\ \nu\dfrac{\partial\mathbf{u}}{\partial\mathbf{n}}-p\mathbf{n}=0 & \text{on } \Gamma_{out} \end{cases}$$

Motivation

$$\begin{cases} -\mu_p \Delta \mathbf{u}(\mu) + \nabla p(\mu) = 0 & \text{in } \Omega_{\mu_G} \\ \nabla \cdot \mathbf{u}(\mu) = 0 & \text{in } \Omega_{\mu_G} \\ \mathbf{u}(\mu) = 0 & \text{on } \Gamma_{D\mu_G} \\ \mu_p \dfrac{\partial \mathbf{u}(\mu)}{\partial \mathbf{n}} - p(\mu)\mathbf{n} = -\mathbf{n} & \text{on } \Gamma_{in\mu_G} \\ \mu_p \dfrac{\partial \mathbf{u}(\mu)}{\partial \mathbf{n}} - p(\mu)\mathbf{n} = 0 & \text{on } \Gamma_{out\mu_G} \end{cases}$$



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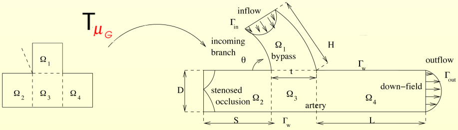
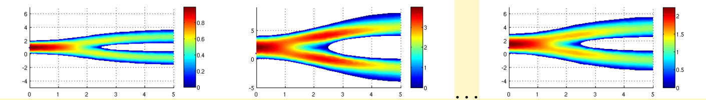


- PDEs are very expensive to solve when solutions need to capture **fine physical details** (velocity boundary layers, wall shear stresses, vorticity layers, etc.)
- Classical numerical techniques (e.g. Finite Element method) are not suitable in a **many query context**, where the solution has to be computed for **many different values of the parameters** (Optimization problem, Multiobjective optimization).
- Reduced order modeling permits to achieve the accuracy and reliability of a **high fidelity approximation** by drastically decreasing the problem complexity and **computational time**.

The Reduced Basis Method

Offline Stage

FEM solutions for a representative set of parameter values $\mu_1, \ldots, \mu_N$ with $N \ll \mathcal{N}$



$u_1^{\mathcal{N}}$ $u_2^{\mathcal{N}}$ $\ldots$ $u_N^{\mathcal{N}}$

The parameter values can be selected by the Greedy algorithm or a optimization greedy algorithm (Volkwein et al.,2012).

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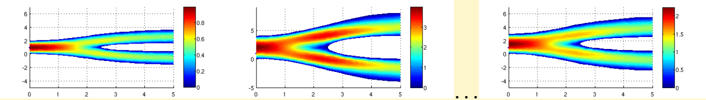

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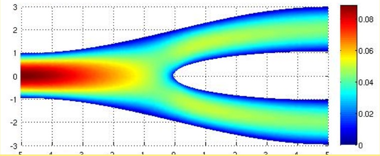
Online Stage

For each new parameter vector μ the RB solution is a weighted combinations of the precomputed solutions (Galerkin projection)



RB method: basic ingredients

High-fidelity discretization technique

(e.g. FEM)

$$A_{\mathcal{N}}(\mu)\mathbf{u}_{\mathcal{N}}(\mu) = \mathbf{f}_{\mathcal{N}}(\mu)$$

$$\|\mathbf{u}(\mu) - \mathbf{u}_{\mathcal{N}}(\mu)\|_X \leq \mathcal{E}(h)$$

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A posteriori error bounds

OFFLINE: reduced space construction (greedy sampling)

$$Z_N = [\mathbf{u}_{\mathcal{N}}(\mu_1)|\ldots|\mathbf{u}_{\mathcal{N}}(\mu_N)]$$

$$\mu^{N+1} = \arg\max_{\mu\in\Xi_{train}} \Delta_N(\mu) \approx \|\mathbf{u}_{\mathcal{N}} - \mathbf{u}_N(\mu)\|_X$$

ONLINE: certification

$$\|\mathbf{u}_{\mathcal{N}}(\mu) - \mathbf{u}_N(\mu)\|_X \leq \Delta_N(\mu)$$

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Galerkin projection

$$Z_N = [\mathbf{u}_{\mathcal{N}}(\mu_1) | \dots | \mathbf{u}_{\mathcal{N}}(\mu_N)]$$

$$\mathbf{u}_{\mathcal{N}}(\mu) \approx Z_N^T \mathbf{u}_N(\mu)$$

$$\underbrace{Z_N^T A_{\mathcal{N}}(\mu) Z_N^T}_{A_N(\mu)} \mathbf{u}_N(\mu) = \underbrace{Z_N^T \mathbf{f}_{\mathcal{N}}(\mu)}_{\mathbf{f}_N(\mu)}$$

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Offline/Online procedure

$$A(\mu) = \sum_{q=1}^{Q_A} \Theta_q^A(\mu) A_q \qquad f(\mu) = \sum_{q=1}^{Q_f} \Theta_q^f(\mu) f_q$$

$$\left(\sum_{q=1}^{Q_A} \Theta_q^A(\mu) A_N^q\right) \mathbf{u}_N(\mu) = \sum_{q=1}^{Q_f} \Theta_q^f(\mu) \mathbf{f}_N^q$$

OFFLINE: expensive database construction

ONLINE: inexpensive evaluation/solution

Multiobjective optimization

- Often in real applications the optimization issue is described by introducing **several objective functions** which compete with each other.
- Optimal solution needs to be taken in the presence of **trade-offs** between two or more conflicting objectives.
- Does not exist a single solution that simultaneously optimizes each objective, but there exists a (possibly infinite) number of **Pareto optimal solutions**.
- The computation of efficient points can be very expensive in particular, if the constraints are given by **partial differential equations**.

PDE-Constrained multiobjective optimization

V and U real, separable Hilbert spaces and $X = V \times U$,
we introduce the vector-valued objective $J : V \times U \to \mathbb{R}^3$ by

$$J_1(x) = \frac{1}{2}\|\mathscr{C}y - w_1\|^2_{W_1} \quad J_2(x) = \frac{1}{2}\|\mathscr{D}y - w_2\|^2_{W_2}, \quad J_3(x) = \frac{\gamma}{2}|u|^2$$

where $x = (y,u) \in X$, $\gamma \geq 0$ W_1, W_2 are Hilbert spaces, $w_1 \in W_1$, $w_2 \in W_2$ and $\mathscr{C}$, $\mathscr{D}$ are linear, bounded from V to W_1 respectively V to W_2.

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Multiobjective problem

to minimize all objectives $J_k, k = 1,2,3$ *"at the same time"*, such that $(y,u) \in V \times U$ solves the linear variational problem

$$a(y,\varphi) = \langle f + \mathscr{B}u, \varphi\rangle_{V',V} \quad \forall \varphi \in V, \tag{1}$$

where $\langle \cdot,\cdot \rangle_{V',V}$ stands for the dual pairing between V and its dual space V' and $\mathscr{B}$ is a continuous, linear operator and $a(\cdot,\cdot)$ a coercive bilinear form.

Multiobjective optimization - Pareto optimal points

First-order necessary conditions for Pareto optimality

Suppose that $\bar{u} \in U$ is Pareto optimal, $\hat{J}_i(u) = J_i(y, u)$. There exists a parameter $\bar{\mu} = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3) \in \mathbb{R}^3$ with $\bar{\mu}_i \geq 0$ and $\sum_{i=1}^3 \bar{\mu}_i = 1$ satisfying

$$\sum_{i=1}^{3} \bar{\mu}_i \hat{J}_i'(\bar{u}) = 0. \tag{2}$$

We define the μ-dependent, scalar-valued objective

$$\hat{\mathfrak{J}}(u; \mu) = \sum_{i=1}^{3} \mu_i \hat{J}_i(u).$$

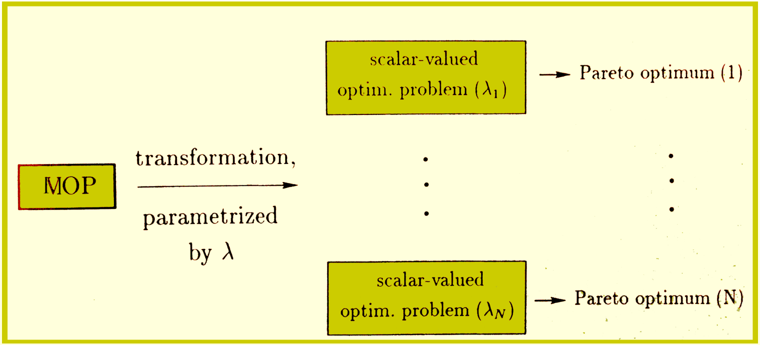
Then, (2) are the necessary optimality conditions for a local solution $\bar{u}$ to

$$\min \hat{\mathfrak{J}}(u; \bar{\mu}) \quad \text{s.t.} \quad (y, u) \in U \times Y \text{ solves (1).} \tag{$\hat{\mathbf{P}}_\mu$}$$

In the weighting method Pareto optimal points are computed by solving ($\hat{\mathbf{P}}_\mu$).

$\star$ Feasible extensions to control constrained and non-linear multiobjective problems.

Multiobjective optimization solved with the RB method



- *C. Hillermeier.* Nonlinear Multiobjective Optimization. A Generalized Homotopy Approach. *Birkhäuser Verlag, Basel, 2001*
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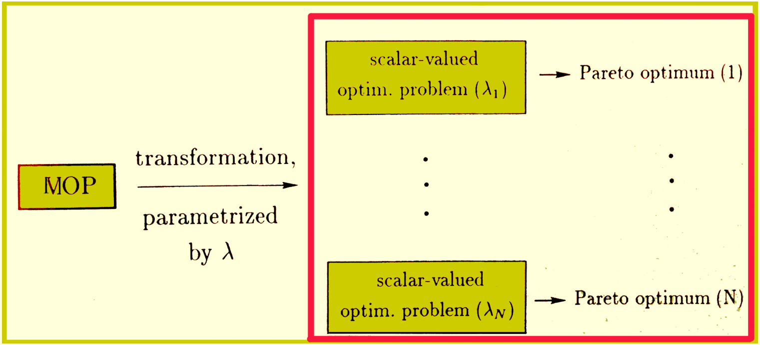

Parametric saddle point formulation solvable by the reduced basis method

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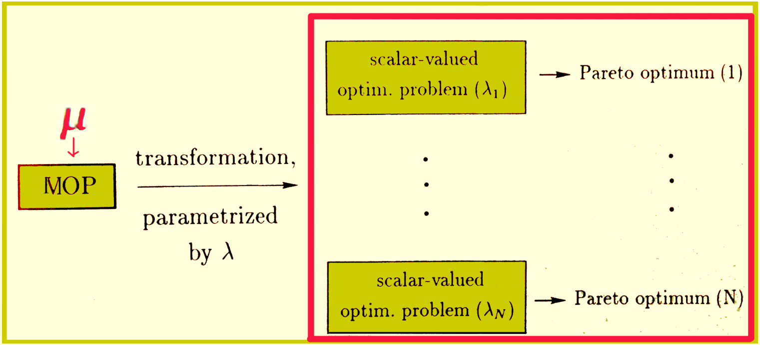

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$$\min \hat{\mathfrak{J}}(u;\bar{\mu}) \quad \text{s.t.} \quad u \in U \text{ solves } a(y,\varphi) = \langle f + \mathscr{C}u, \varphi \rangle$$

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Saddle-point formulation (Brezzi theorem for existence and uniqueness).

$$\begin{cases} \mathscr{A}(x(\mu), w; \mu) + \mathscr{B}(w, p(\mu); \mu) = \langle \mathscr{F}(\mu), w \rangle, & \forall w \in X \\ \mathscr{B}(x(\mu), \varphi; \mu) = \langle f, \varphi \rangle, & \forall \varphi \in Q \end{cases} \tag{3}$$

$p(\mu)$ is the Lagrange multiplier associated to the constrain.

$$\mathscr{L}(x,p;\mu) = \mathfrak{J}(x,\mu) + \mathscr{B}(x,p;\mu) - \langle f, p \rangle.$$

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Multiobjective optimization solved with the RB method

RB spaces

Snapshots:

$\{y^{\mathcal{N}}(\mu^n), n = 1, \ldots, N\}$, $\{u^{\mathcal{N}}(\mu^n), n = 1, \ldots, N\}$,$\{p^{\mathcal{N}}(\mu^n), n = 1, \ldots, N\}$

State and adjoint space: $Z_N = span\{y^{\mathcal{N}}(\mu^n), p^{\mathcal{N}}(\mu^n), n = 1, \ldots, N\}$.

Control space: $U_N = span\{u^{\mathcal{N}}(\mu^n), n = 1, \ldots, N\}$

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A posteriori error estimates

A posteriori error bound for the solution:

$$(||x^{\mathcal{N}}(\mu) - x_N(\mu)||_X^2 + ||p^{\mathcal{N}}(\mu) - p_N(\mu)||_Q^2)^{1/2} \le \Delta_N(\mu) = \frac{||r(\cdot,\mu)||_{\mathscr{X}'}}{\beta^{\mathcal{N}}(\mu)}$$

A posteriori error bound for the cost functional:

$$|J(y^{\mathcal{N}}(\mu), u^{\mathcal{N}}(\mu);\mu) - J(y^N(\mu), u^N(\mu);\mu)| \le \Delta_N^J(\mu) = \frac{1}{2}\frac{||r(\cdot,\mu)||_{\mathscr{X}'}^2}{\beta^{\mathcal{N}}(\mu)}$$

$||r(\cdot,\mu)||_{\mathscr{X}'}$ dual norm of the residual

$\beta^{\mathcal{N}}(\mu)$ Babuska inf-sup constant (or a proper lower bound).

Sensitivity analysis

Let consider $\hat{\mu}$ as first parameter to compute the first Pareto point, we are interested in choosing the next μ such that the weights μ_1 and μ_2 lead to significant changes of the cost functional

$$\hat{\mathfrak{J}}(u;\hat{\mu}) = \frac{\hat{\mu}_1}{2}\,\|\mathscr{C}\hat{y} - w_1\|^2_{W_1} + \frac{\hat{\mu}_2}{2}\,\|\mathscr{D}\hat{y} - w_2\|^2_{W_2} + (1-\hat{\mu}_1-\hat{\mu}_2)\,\frac{\gamma}{2}\,|\hat{u}|^2.$$

Taylor expansion of the reduced objective with respect to changes in μ_1 and μ_2 is:

$$\tilde{\mathfrak{J}}(u;\mu) = \hat{\mathfrak{J}}(\hat{u};\hat{\mu}) + \frac{\partial\hat{\mathfrak{J}}}{\partial\mu_1}(\hat{u};\hat{\mu})(\mu_1-\hat{\mu}_1) + \frac{\partial\hat{\mathfrak{J}}}{\partial\mu_2}(\hat{u};\hat{\mu})(\mu_2-\hat{\mu}_2)$$

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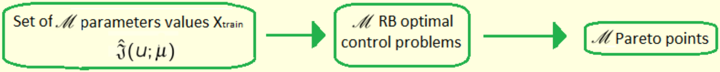

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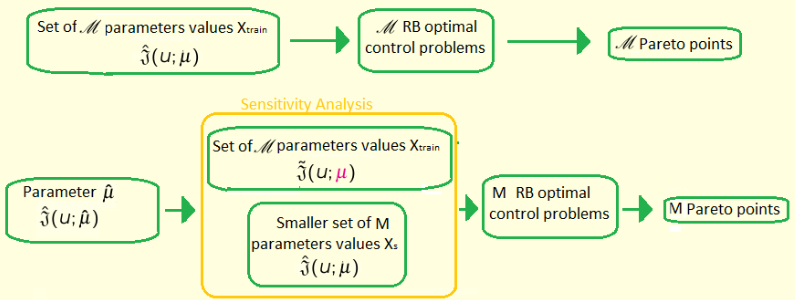

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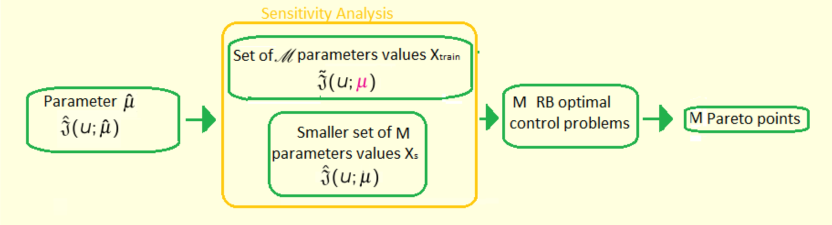
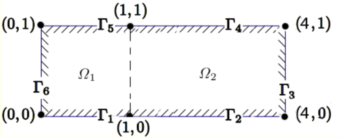


- RB offline step for the RB approximation of the saddle point problem;
- RB online step for defining the RB optimal solutions corresponding to initial parameter guesses;
- sensitivity analysis for defining a suitable parameter set X_s that leads to significant variations of the cost functional;
- RB Pareto optimal solutions for the parameter set X_s.

The sensitivity analysis allows to drastically reduce the number of online RB computations needed to recover a suitable distribution of the Pareto optimal solutions.

Numerical results

We consider the domain Ω given by a rectangle separated in two subdomains Ω_1 and Ω_2.



$$J_1(y) = \frac{1}{2}||y - y_d||^2_{L^2(\Omega)}, \quad J_2(y) = \frac{1}{2}||\nabla y||^2_{L^2(\Omega)}, \quad J_3(u) = \frac{\alpha}{2}||u||^2_U$$

where $y_d = 1$ in Ω_1 and $y_d = 0.6$ in Ω_2. The state function $y \in Y = H^1_0(\Omega)$ solves the following Laplace problem:

$$\begin{cases} -\Delta y = u & \text{in } \Omega, \\ y = 1 & \text{on } \Gamma_D = \partial\Omega, \end{cases} \tag{4}$$

where $u \in L^2(\Omega)$ is the control function.
In order to apply the Pareto optimal theory we introduce the following cost functional:

$$\hat{J}(y(\mu), u(\mu), \mu) = \mu_1 J_1(y(\mu)) + \mu_2 J_2(y(\mu)) + (1 - \mu_1 - \mu_2) J_3(u(\mu)),$$

and the **parametrized optimal control problem**:

$$\min_{y,u} \hat{J}(y(\mu), u(\mu), \mu) \text{ s. t. } (y(\mu), u(\mu)) \in Y \times U \text{ solves (4).}$$

Numerical results

The numerical approximation of the reduced basis functions (state, control and adjoint variables) is computed by using $\mathbb{P}_1$ finite elements. $\mathscr{N} = 11441$. $\mu_1 \in [0,1]$ and $\mu_2 \in [0, 1-\mu_1]$.

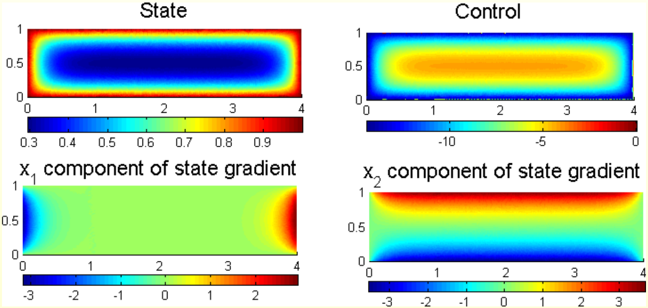


Figure: Representative solution for $\mu = (0.11, 0.83)$.

Numerical results

The numerical approximation of the reduced basis functions (state, control and adjoint variables) is computed by using $\mathbb{P}_1$ finite elements. $\mathcal{N} = 11441$. $\mu_1 \in [0,1]$ and $\mu_2 \in [0, 1-\mu_1]$.

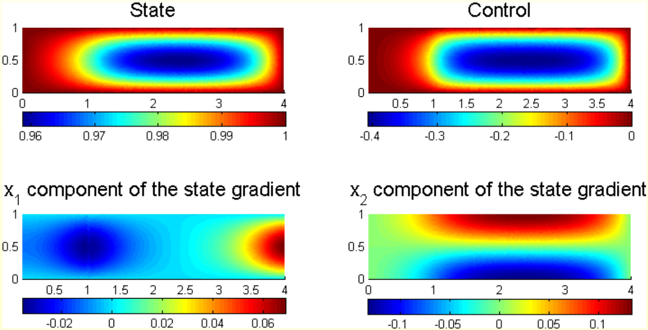


Figure: Representative solution for $\mu = (0.9, 0)$.

Numerical results

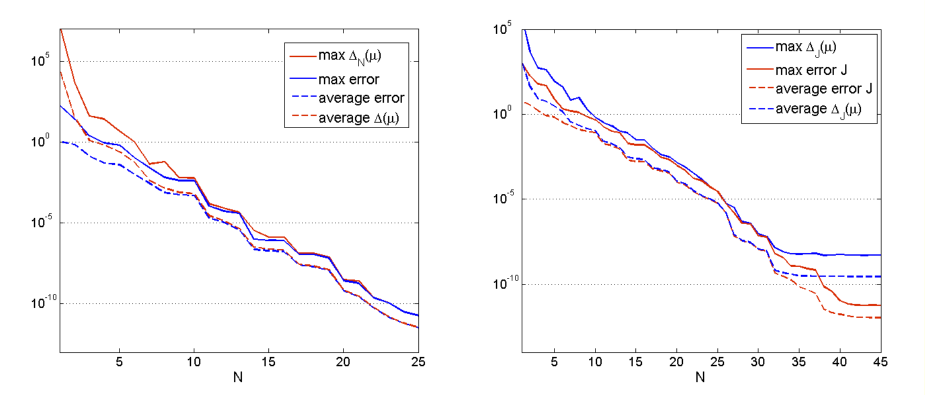


Figure: Average and maximum errors and error bounds regarding the solution of the problem (left) and the cost functional (right) between the FE and RB approximations.

Numerical results

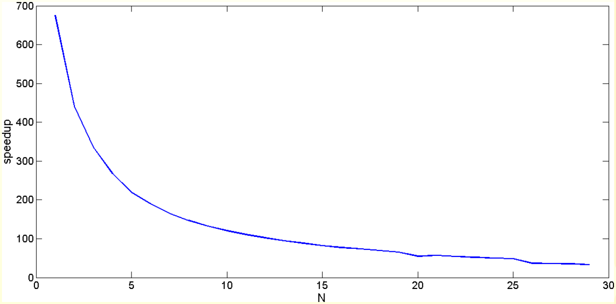


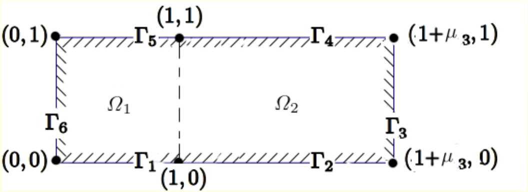
Figure: speedup with respect to a FE computational time by varying the number of basis functions.

Offline CPU time: 21 minutes for selecting the RB bases.
Online evaluation time (15 basis functions-45 in total for state, control and adjoint) and including the evaluation of the a posteriori error bound is 0.016 seconds.

Evaluation of the FE solution requires about 1.26 seconds, a speedup equal to 88,32.

Numerical results

We consider a smaller control space, $U = \text{span}\{b_1, b_2\}$ where b_1 and $b_2 \in L^\infty(\Omega)$ are the characteristic functions of Ω_1 and Ω_2 respectively. We introduce a geometrical parameter $\mu_3 \in [1, 3.5]$ that defines the length of the domain Ω.



We recall the vector cost functional defined as follows:

$$J_1(y) = \frac{1}{2}||y - y_d||^2_{L^2(\Omega)}, \quad J_2(y) = \frac{1}{2}||\nabla y||^2_{L^2(\Omega)}, \quad J_3(u) = \frac{\alpha}{2}||u||^2_U \tag{5}$$

where $y_d = 1$ in Ω_1 and $y_d = 0.6$ in Ω_2. The state function $y \in Y = H^1_0(\Omega)$ solves the following Laplace problem:

$$\begin{cases} -\Delta y = \sum_{i=1}^2 u_i b_i & \text{in } \Omega_\mu, \\ y = 1 & \text{on } \Gamma_D = \partial\Omega_\mu, \end{cases} \tag{6}$$

where $u_i \in \mathbb{R}^2$ define the control function.

Numerical results

$$S = \{(w_1, w_2) \in \mathbb{R}^2 : w_1 = J_1(u), w_2 = J_2(u), u = \sum_{i=1}^{2} u_i b_i, -30 \leq u_i \leq 10\}, \quad \mu_3 = 3.$$

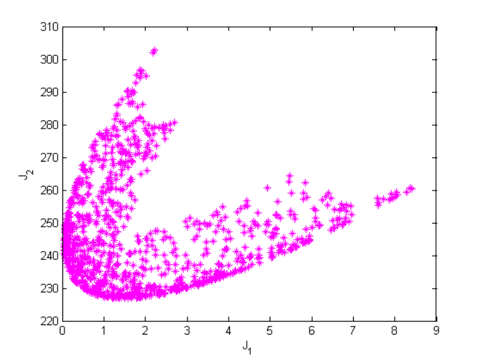


Figure: Set of the possible values of the cost functionals J_1 and J_2, by varying the function u.

Numerical results

$$S = \{(w_1, w_2) \in \mathbb{R}^2 : w_1 = J_1(u), w_2 = J_2(u), u = \sum_{i=1}^{2} u_i b_i, -30 \le u_i \le 10\},$$

Pareto points $\mu_1, \mu_2 \in [0,1], \mu_3 = 3$

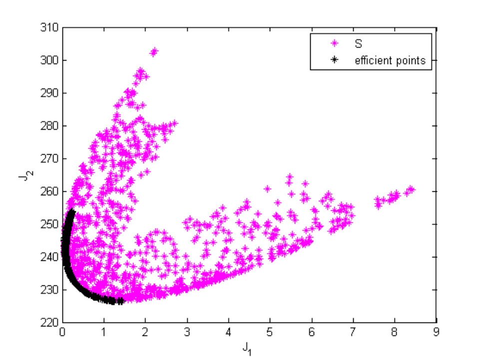


Figure: Set of the possible values of the cost functionals J_1 and J_2, by varying the function u and the subset of the efficient points.

Numerical results

$$S = \{(w_1, w_2, w_3) \in \mathbb{R}^2 : w_1 = J_1(u), w_2 = J_2(u), w_3 = J_3(u) u = \sum_{i=1}^{2} u_i b_i, -30 \le u_i \le 10\}$$

Pareto points $\mu_1, \mu_2 \in [0, 1], \mu_3 = 3$

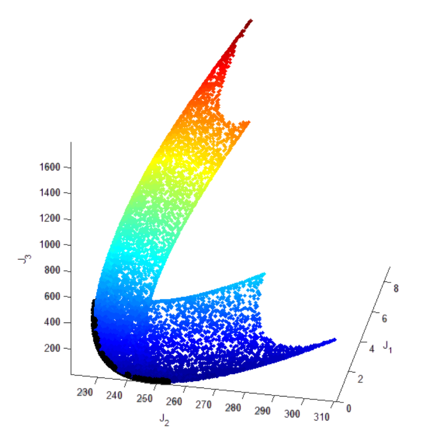
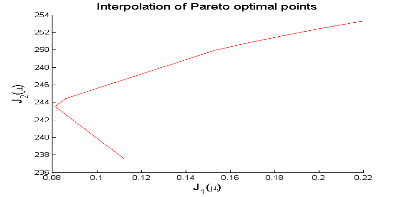


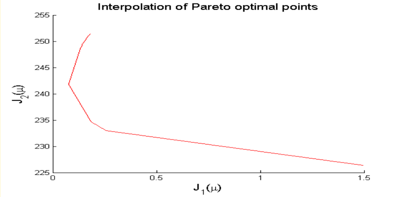
Figure: Feasible points considering J_1, J_2, J_3 by varying u and the subset of the efficient points.

Sensitivity analysis

10 Random solutions

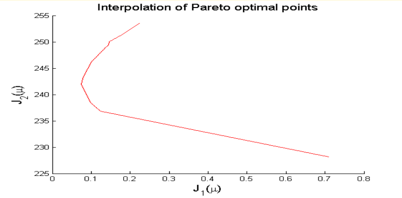


10 Sensitive solutions

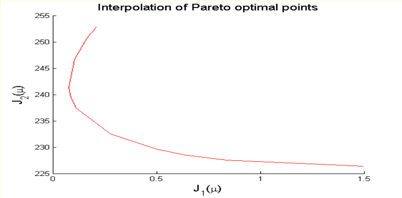


Sensitivity analysis

20 Random solutions

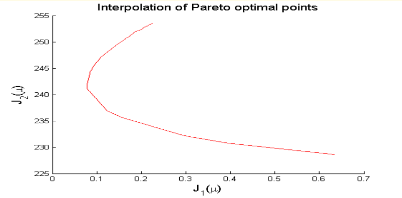


20 Sensitive solutions

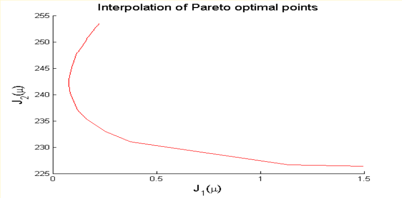


Sensitivity analysis

30 Random solutions

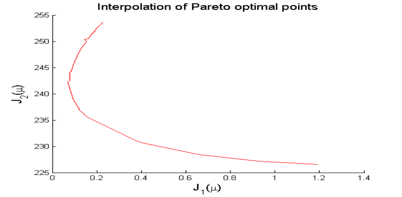


30 Sensitive solutions

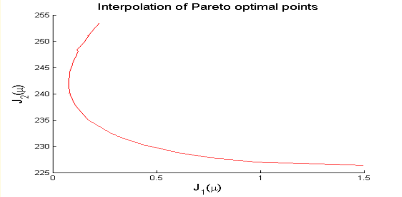


Sensitivity analysis

60 Random solutions

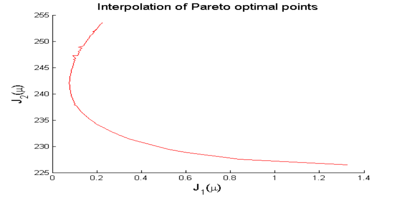


60 Sensitive solutions

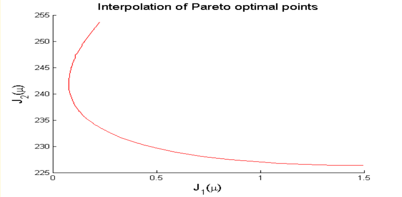


Sensitivity analysis

100 Random solutions

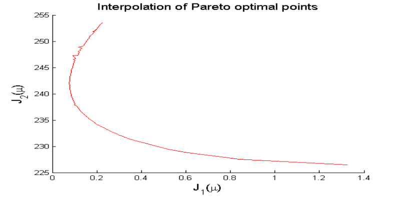


100 Sensitive solutions

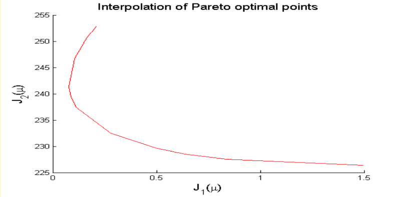


Sensitivity analysis

100 Random solutions



20 Sensitive solutions



Conclusions

- Model order reduction strategy proposed for solving the **multiobjective optimal problems**, characterized by more than one cost functional.
- The multiobjective problem leads to a **parametric optimal control problem** through a new parametric cost functional defined as a **weighted sum of the original cost functionals**.
- The **Pareto optimal points** are the optimal controls corresponding to the problem considering a different weighted sum of the cost functionals.
- The use of the **RB method**, together with an useful and inexpensive **sensitive analysis**, allows to drastically reduce the computational times compared with FE.
- A **rigorous error bound analysis** permits to ensure a certain level of accuracy of the solution.

THANK YOU FOR YOUR ATTENTION!