# A parametric ROM for the linear frequency domain approach to time-accurate CFD

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# A parametric ROM for the LFD approach to CFD

- **Motivation: Parametric reduced-order models**
- Interpolating data on manifolds
- The Linear Frequency Domain (LFD) approach to CFD
- Offline, semi-offline and online ROM stages
- A practical example
- Conclusions & Outlook

# Motivation: Parametric reduced-order models

**Key idea of subspace based reduced order modeling:**

- Construct **low-dimensional** subspace
- Restrict computations to **low-dimensional** subspace (e.g. via projection)

**Examples:**

- Reduced Bases methods
- Krylov subspace methods
- Proper Orthogonal Decomposition

# Motivation: Parametric reduced-order models

**In this talk:**

**Parameter dependent large-scale linear systems:**

$$A(p,q)W = b(p,q), \quad A \in \mathbb{R}^{n\times n}, \; b \in \mathbb{R}^n$$

- Parameters $q$ determine operating points
- Parameters $p$ are considered as examination parameters

## Motivation: Parametric reduced-order models

**Standard order reduction ansatz:** use subspace of dim $m \ll n$

$$W(p, q_0) = U_{q_0} a(p), \quad U_{q_0} \in \mathbb{R}^{n \times m}$$

Determine coefficient vector $a(p) \in \mathbb{R}^m$ via

- **Orthogonal residual condition**

$$U_{q_0}{}^T S A U_{q_0} a = U_{q_0}{}^T S b \in \mathbb{R}^m$$

- **Minimum residual condition**

$$U_{q_0}{}^T A^T S A U_{q_0} a = U_{q_0}{}^T A^T S b \in \mathbb{R}^m$$

inner product: $\langle \cdot, \cdot \rangle = (\cdot)^T S(\cdot)$

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$$\boxed{A_{orth} a = b_{orth}}$$

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$$\boxed{A_{\min} a = b_{\min}}$$

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inner product: $\langle \cdot , \cdot \rangle = (\cdot)^T S (\cdot)$

## Motivation: Parametric reduced-order models

- Suppose that at r operating points, we have computed bases

$$U_{q_1}, \dots, U_{q_r}$$

- Objective: Construct ROM at new operating point

$$W(p, q_*) = U_{q_*} a(p), \quad U_{q_*} \in \mathbb{R}^{n \times m}$$

- Required: Parametrized **trajectory** of projection bases

$$U: q \longmapsto U_q$$

- Idea: **Interpolate** given bases [1] [2]

[1] D. Amsallem, C. Farhat, 2008

[2] Nguyen T.S., 2012

# Motivation: Parametric reduced-order models

**Challenges:**

- ROM subspace spanned by orthogonal reduced-order basis (ROB)
- Ensure that parametric variation always gives orthogonal bases

Reduced bases matrices **must** be considered as points in the Grassmann manifold of k-dimensional subspaces or the Stiefel manifold of “tall-skinny” orthonormal matrices[3]

[3] A. Edelman, T.A. Arias, S.T. Smith, 1998

# A parametric ROM for the LFD approach to CFD

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# Interpolating data on manifolds (A): basic principle

## Matrix manifolds

- **Stiefel manifold**

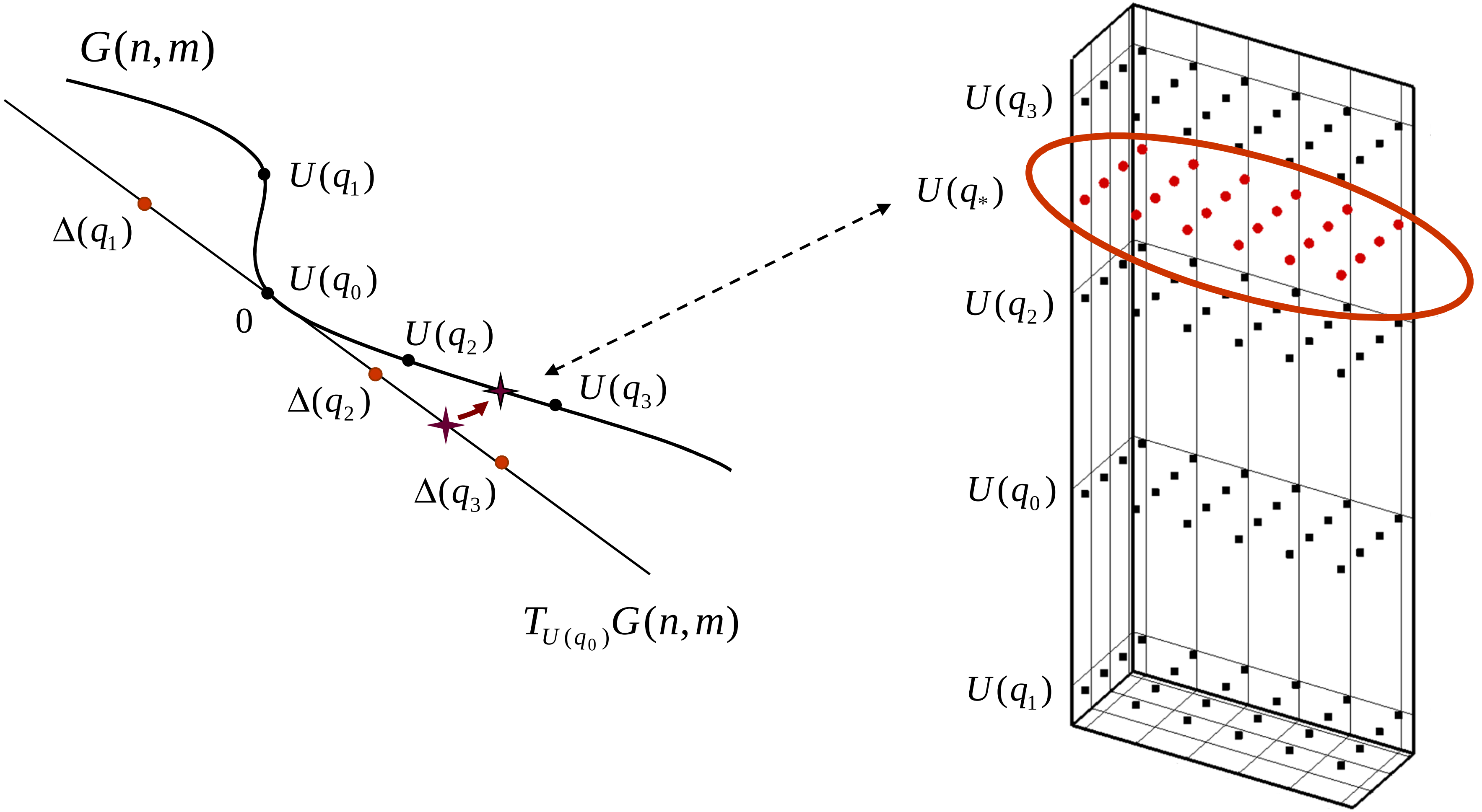
$$V(n,m) = \{Y \in \mathbb{R}^{n\times m} |\ Y^T Y = I_{m\times m}\}$$

- **Grassmann manifold**

$$G(n,m) = \{S \leq \mathbb{R}^{n} | \dim(S) = m\}$$

=> each point is a subspace of dimension *m*,
equivalence class of matrices spanning a given subspace may be represented by point in Stiefel manifold

# Interpolating data on manifolds (A): basic principle



# Interpolating data on manifolds (B): Matrix Manifolds

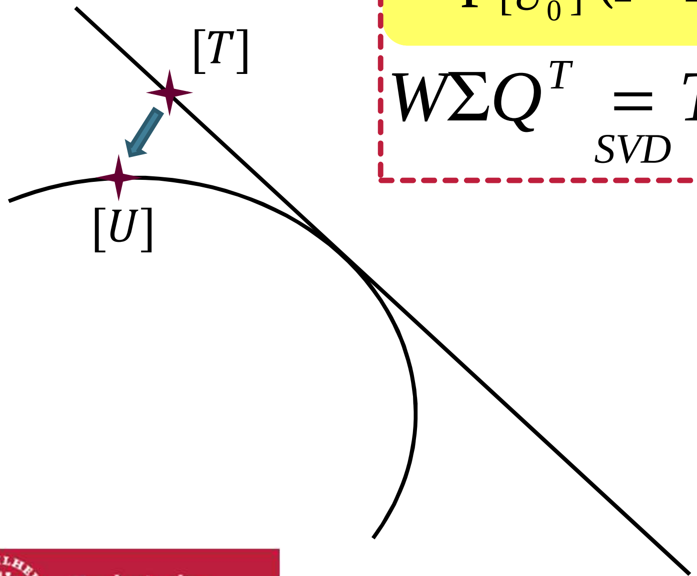
**…and here numerical linear algebra comes in**

- **Grassmann exponential mapping**

$$[T] \in T_{[U_0]}G(n,m), \quad U_0^T T = 0_{m\times m}:$$

$$\exp_{[U_0]}\left([T]\right) = [U_0 Q\cos(\Sigma) + W\sin(\Sigma)] \in G(n,m),$$

$$W\Sigma Q^T \underset{SVD}{=} T$$

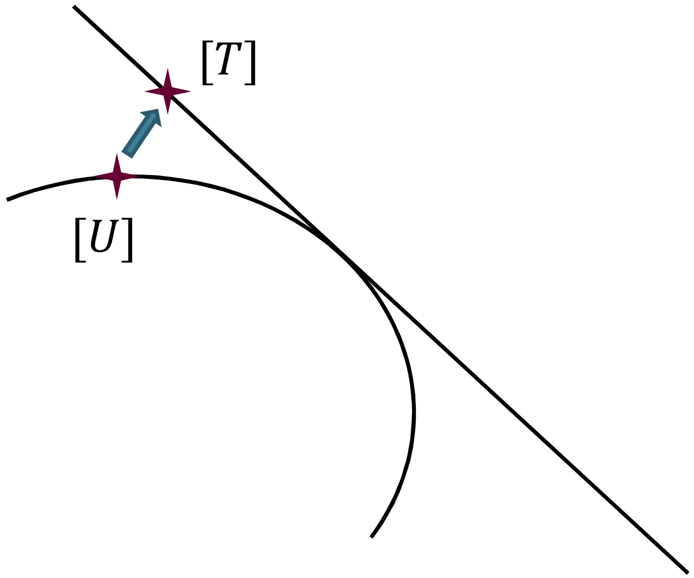



Isometric mapping: tangent space to manifold

# Interpolating data on manifolds (B): Matrix Manifolds

**…and here numerical linear algebra comes in**

- **Grassmann logarithmic mapping**




$$[U] \in G(n,m), \quad U^T U = I_{m \times m}:$$

$$\log_{[U_0]}([U]) = [T] \in T_{[U_0]} G(n,m),$$

$$T = W \tan^{-1}(\Sigma),$$

$$W \Sigma Q^T \underset{SVD}{=} (I_{n \times n} - U_0 U_0^T) U (U^T U_0)^{-1}$$

Isometric mapping: manifold to tangent space

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# Interpolating data on manifolds (C): A word of caution

**Challenges**

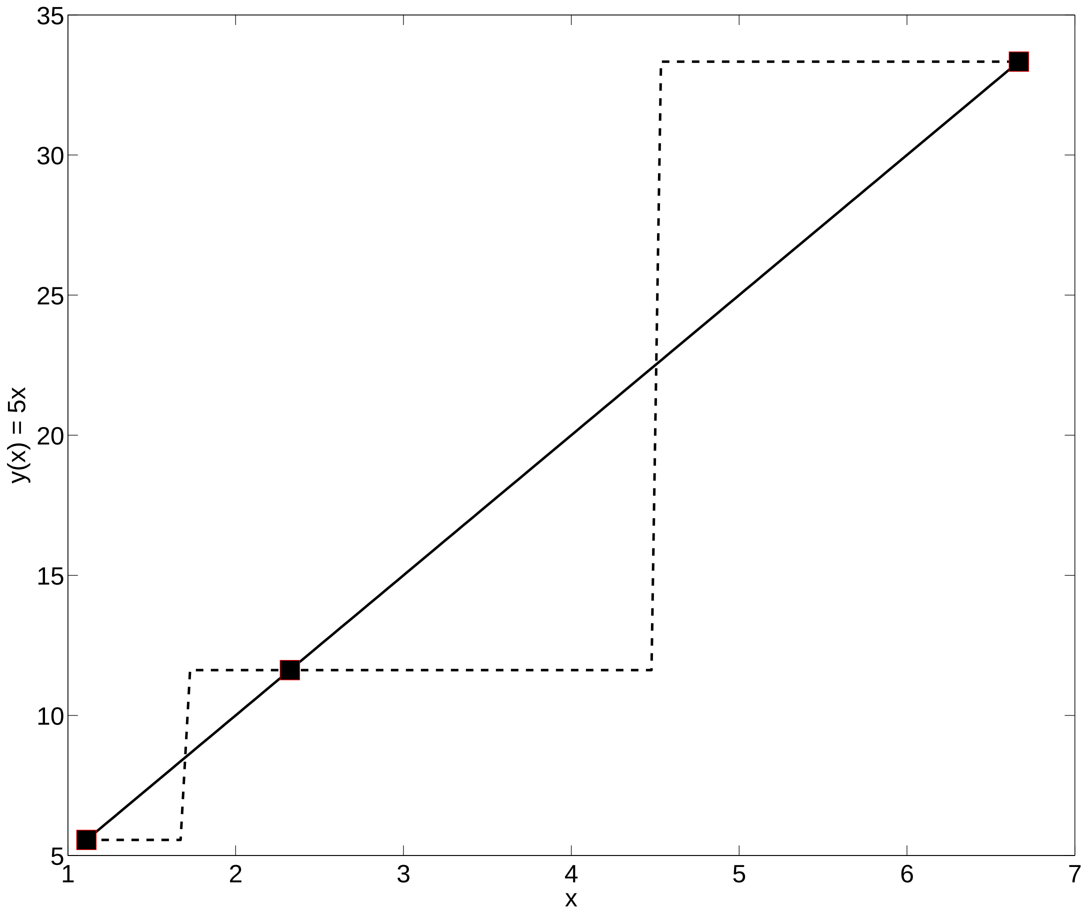
- It cannot be taken for granted that the interpolation's result stays in the tangential space!
  - There are interpolation schemes, that are not admissible in this regard!

- Tangential vectors are represented by $(n \times m)$-matrices:
  - Entry-by-entry interpolation may require millions of interpolation procedures

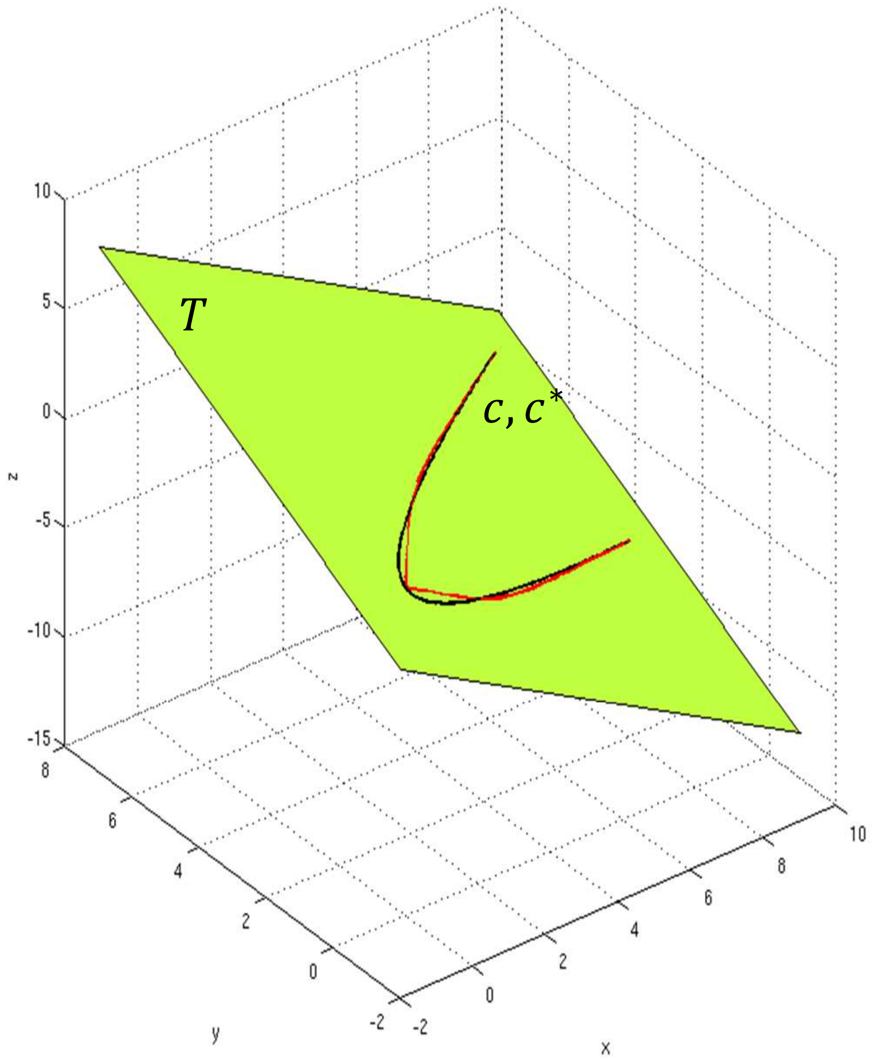
# Interpolating data on manifolds (C): A word of caution

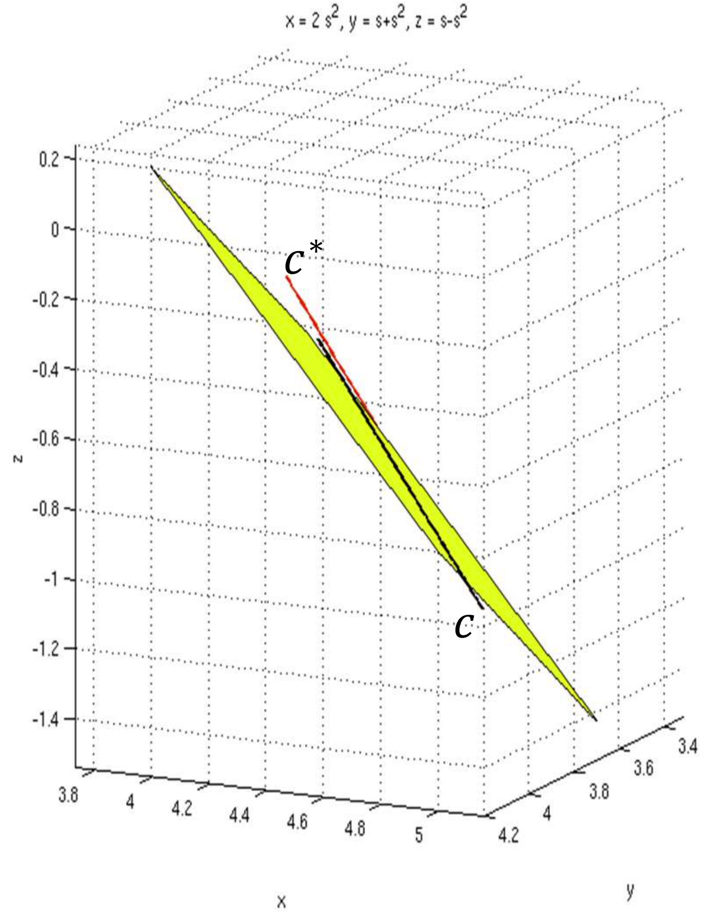
Trivial example:

Nearest-neighbor interpolation leaves the data subspace



# Interpolating data on manifolds (C): A word of caution



## Interpolating data on manifolds (C): A word of caution

**Conclusion**

- Let $T^1 = T(q_1), \dots, T^k = T(q_k)$ be vectors in a Grassmann tangent space, represented by $n \times m$ – matrices
- An admissible interpolator must be of the form

$$T^*(q) = \sum_{l=1}^{k} f_l(q) T^l$$

- In 1D, e.g. **linear** interpolation and **Lagrange** interpolation fulfill this requirement
- In multiple D, **radial basis function** interpolation is feasible

# A parametric ROM for the LFD approach to CFD

- Motivation: Parametric reduced-order models
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- **The Linear Frequency Domain (LFD) approach to CFD**
- Offline, semi-offline and online ROM stages
- A practical example
- Conclusions & Outlook

## The linear frequency approach to CFD

**Underlying assumption**

- Flow to be simulated is **periodic** and approximately **harmonic**
  - Valid for pitching motions of small amplitude

- Flow state vector given by **mean flow** + **time-dependent fluctuations**

$$W(t) = \bar{W} + \tilde{W}(t) \in \mathbb{R}^N, \quad N = \#(\text{flow vars})\,\#(\text{grid points})$$

- Same applies to **grid coordinates** and **grid cell volumes**

# The linear frequency approach to CFD

**First-order Taylor and transition to Fourier space give complex linear system:**

$$(DR_W + i\omega\bar{M})\widehat{W} = b_1 + i\omega b_2 \in \mathbb{C}^N$$

($\bar{M}$ = temporal mean of grid cell volumes, $\omega$ = frequency)

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# The linear frequency approach to CFD

**First-order Taylor and transition to Fourier space give complex linear system:**

$$(DR_W + i\omega\bar{M})\widehat{W} = b_1 + i\omega b_2 \in \mathbb{C}^N$$

$$A(\omega, Ma)\widehat{W} = b(\omega, Ma) \in \mathbb{C}^N$$

($\bar{M}$ = temporal mean of grid cell volumes, $\omega$ = frequency)

**Parametric dependency:**

- Mach number determines operating point (mean value) $q = Ma$
- Frequency varies at each operating point $p = \omega$

$$\widehat{W} = \widehat{W}(\omega, Ma) \in \mathbb{C}^N$$

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# Offline, semi-offline and online stage

## Offline stage

- Select operating points (**OPs**) $q_1, \dots, q_r$
- At each **OP**, compute projection basis $U_{q_k}$ via **snapshot POD**

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## Offline, semi-offline and online stage

### Offline stage

- Select operating points (**OPs**) $q_1, \ldots, q_r$
- At each **OP**, compute projection basis $U_{q_k}$ via **snapshot POD**

- For any **OP** $q_*$ of interest, assess the trajectory

$$U: q \longmapsto U_q$$

via Grassmann Interpolation to obtain $U_{q_*} = U(q_*) \in \mathbb{C}^{N \times m}$

# Offline, semi-offline and online stage

**Semi-offline stage**

- At operating point $q_*$, project full order system onto $U_{q_*}$
- Exploit **separable** parameter dependency:

$$A_{min}(\omega, q_*) = {U_{q_*}}^T (DR_W + i\omega\bar{M})^T S (DR_W + i\omega\bar{M}) U_{q_*} = E(q_*) + \omega^2 G(q_*) + i\omega H(q_*) \in \mathbb{C}^{m\times m}$$

$$b_{min}(\omega, q_*) = \beta_{min,1}(q_*) + \omega^2 \beta_{min,2}(q_*) + i\omega\beta_{min,3}(q_*) \in \mathbb{C}^m$$

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## Offline, semi-offline and online stage

**Semi-offline stage**

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$$A_{min}(\omega, q_*)$$
$$= {U_{q_*}}^T (DR_W + i\omega\bar{M})^T S (DR_W + i\omega\bar{M}) U_{q_*}$$
$$= E(q_*) + \omega^2 G(q_*) + i\omega H(q_*) \in \mathbb{C}^{m\times m}$$

Similar for **orth res** approach

$$b_{min}(\omega, q_*)$$
$$= \beta_{min,1}(q_*) + \omega^2 \beta_{min,2}(q_*) + i\omega \beta_{min,3}(q_*) \in \mathbb{C}^m$$

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# Offline, semi-offline and online stage

## Semi-offline stage

- At operating point $q_*$, project full order system onto $U_{q_*}$
- Exploit **separable** parameter dependency: precompute

$$E(q_*), \qquad G(q_*), \qquad H(q_*) \in \mathbb{C}^{m\times m}$$

$$\beta_{min,1}(q_*), \;\; \beta_{min,2}\,(q_*), \;\; \beta_{min,3}(q_*) \in \mathbb{C}^{m}$$

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## Offline, semi-offline and online stage

### Online stage

- For any frequency $\omega$ compute reduced system and right hand side

$$A_{min}(\omega, q_*) = E(q_*) + \omega^2 G(q_*) + i\omega H(q_*) \in \mathbb{C}^{m\times m}$$

$$b_{min}(\omega, q_*) \\ = \beta_{min,1}(q_*) + \omega^2 \beta_{min,2}(q_*) + i\omega\beta_{min,3}(q_*)$$

- Solve the (m x m) system

$$A_{min}(\omega, q_*)a = b_{min}(\omega, q_*)$$

- Effort: $O(m^3)$ independent of full scale dimension = **real time online stage!**

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# A practical example

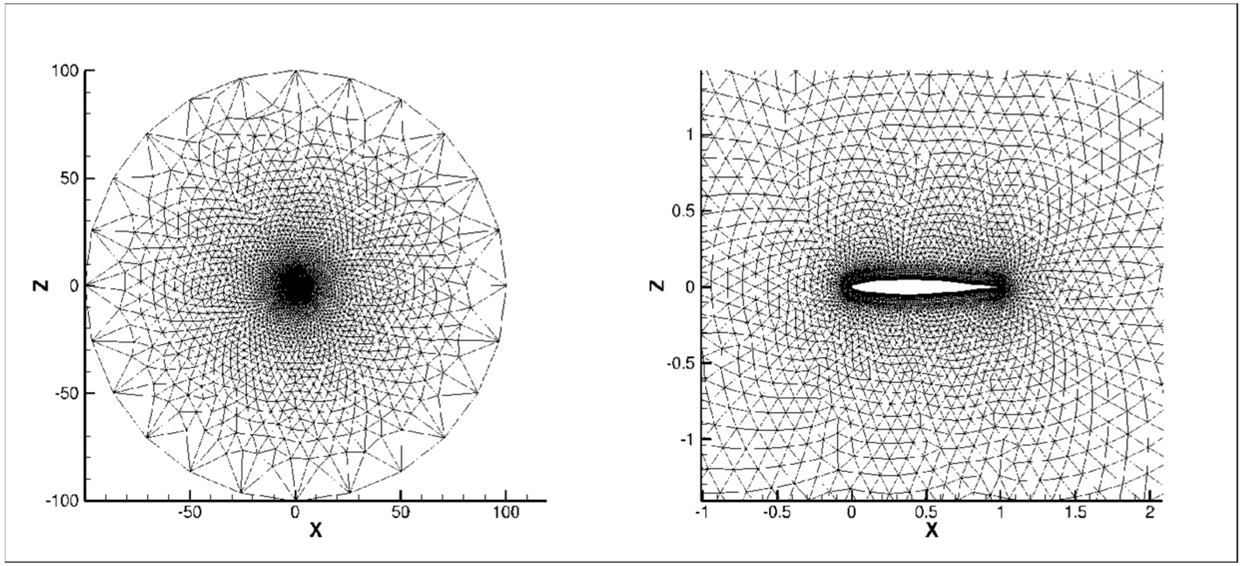
## Grassmann interpolation of ROBs for LFD approach to CFD

**NACA64A010 airfoil:**

Snapshots in transonic

Regime

**Shocks**




Computational grid, size: 10,727 points

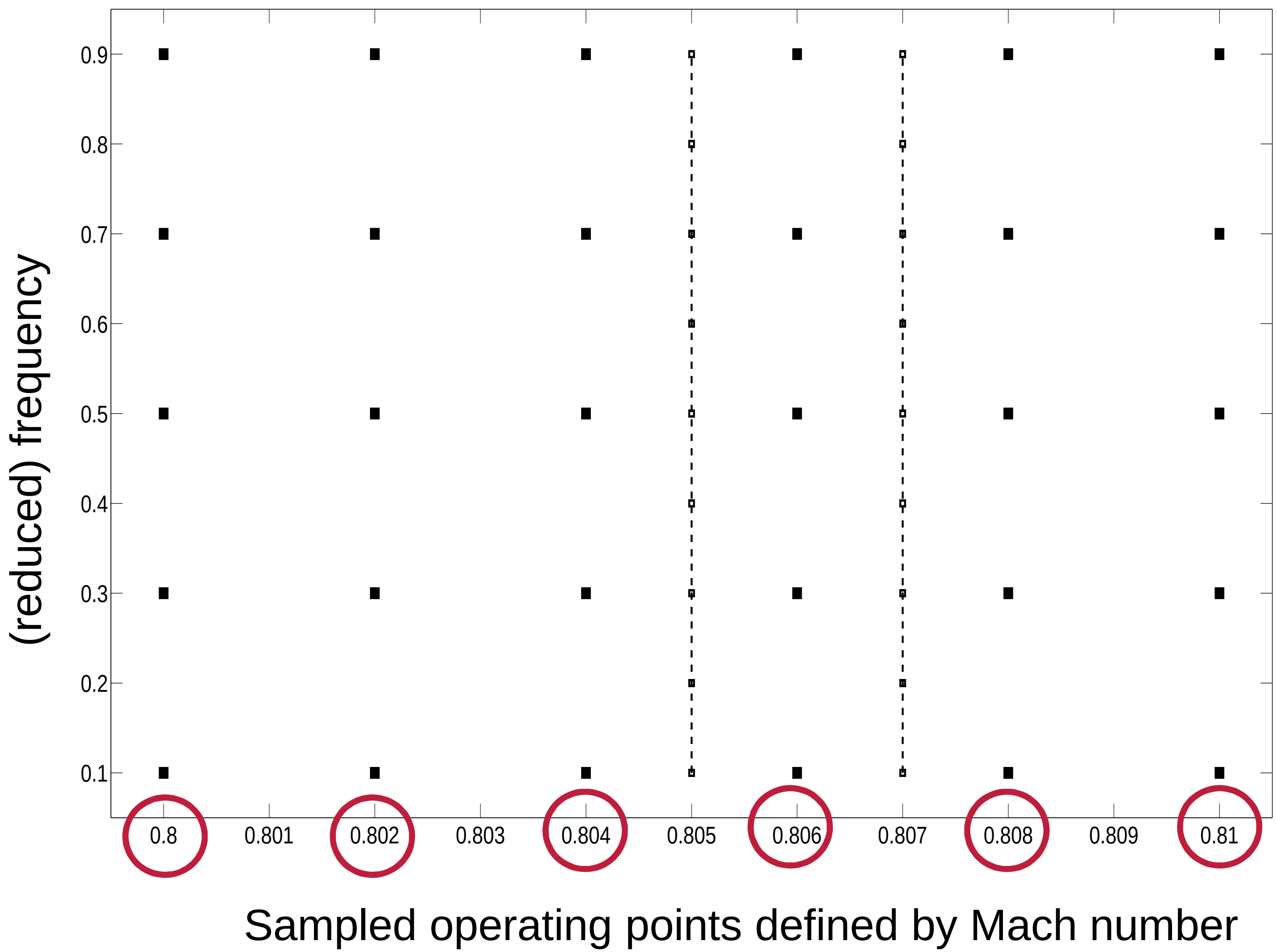
2-parameter model:

$$Ma \in \{0.80,\ 0.802,\ 0.804,\ 0.806, 0.808, 0.81\}, \quad \kappa \in \{0.1,\ 0.3,\ 0.5,\ 0.7,\ 0.9\}$$

$$\overline{\alpha} \in 0^{\circ}, \ \ \mathrm{Re} \in 7.5 mio \text{ fixed}$$

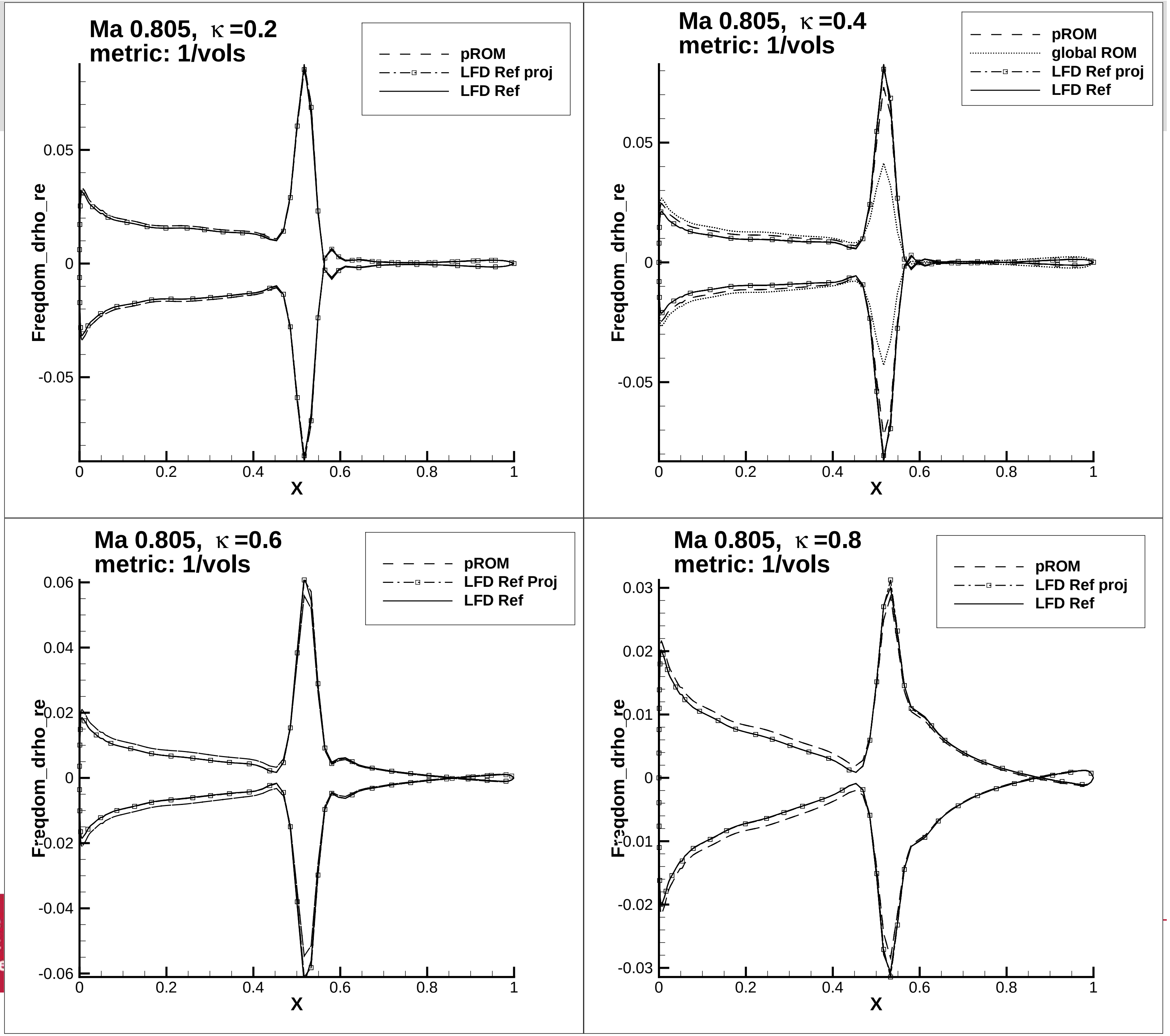
# A practical example

**Snapshot and OP-bases sample plan**



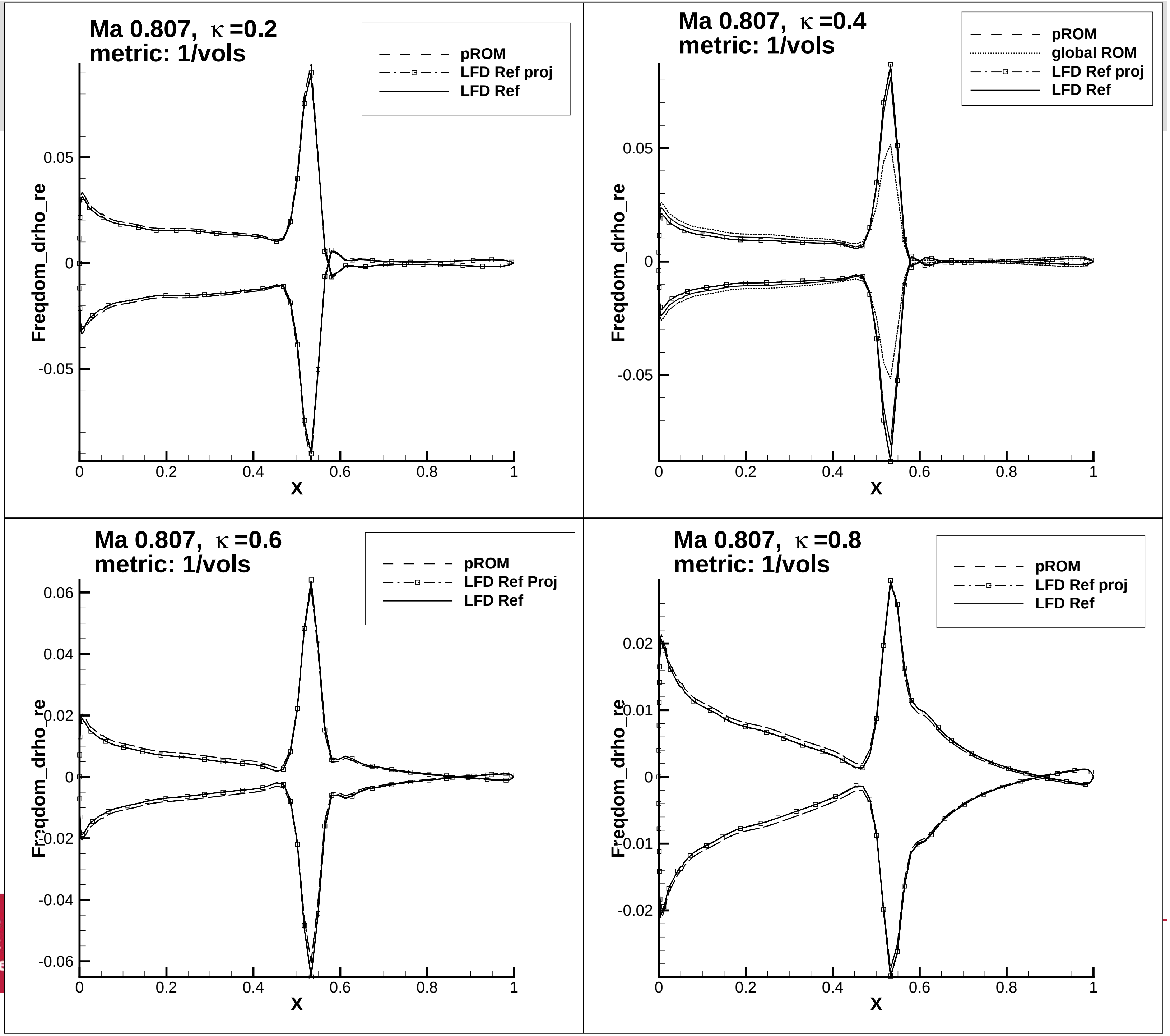
# A practical example

$$\langle \cdot,\cdot \rangle = (\cdot)^T \overline{M}^{-1} (\cdot)$$



A practical example

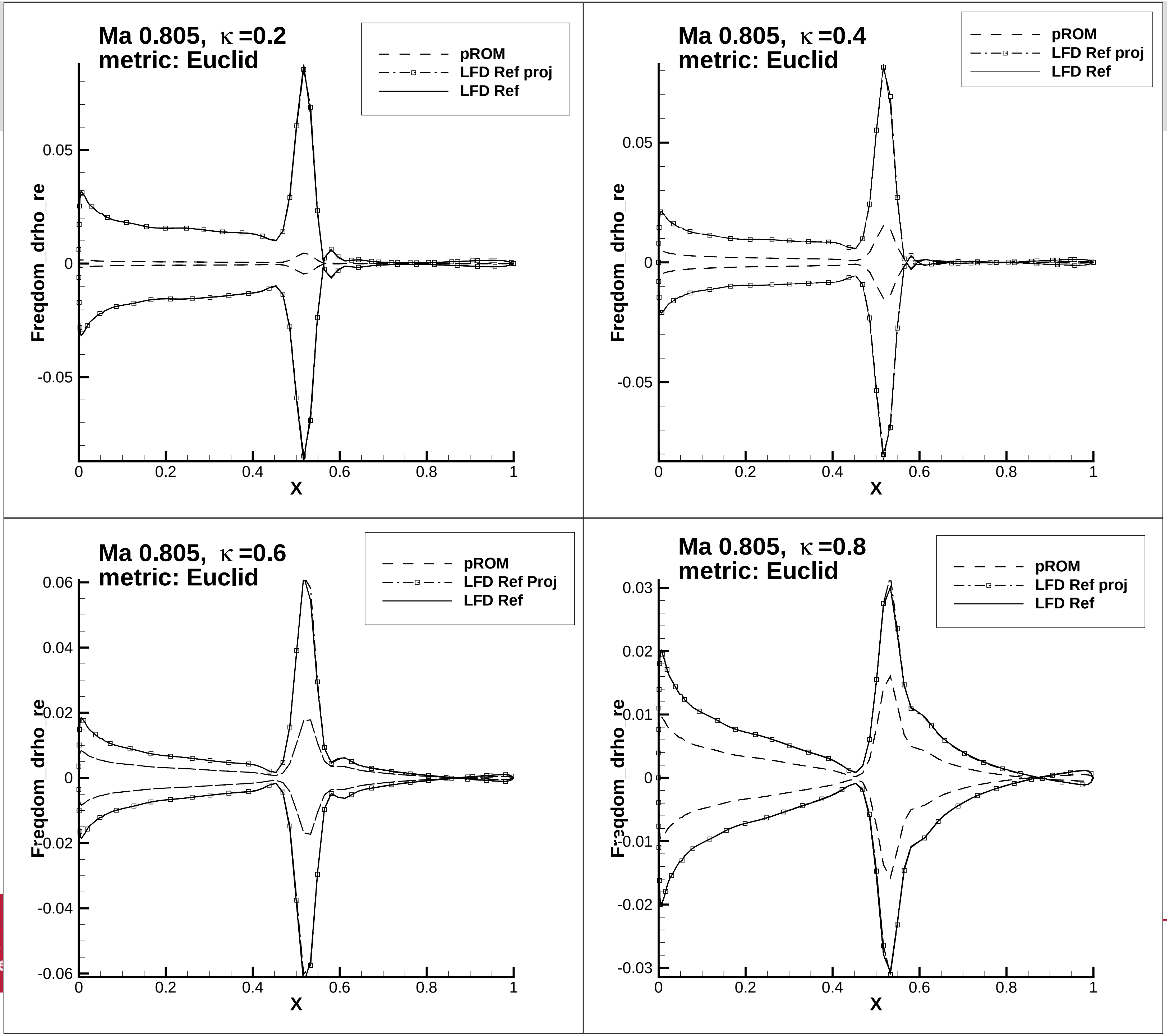
$$\langle \cdot , \cdot \rangle = (\cdot)^T \overline{M}^{-1} (\cdot)$$



A practical example

**Metrics matter!**

$$\langle \cdot , \cdot \rangle = (\cdot)^T (\cdot)$$

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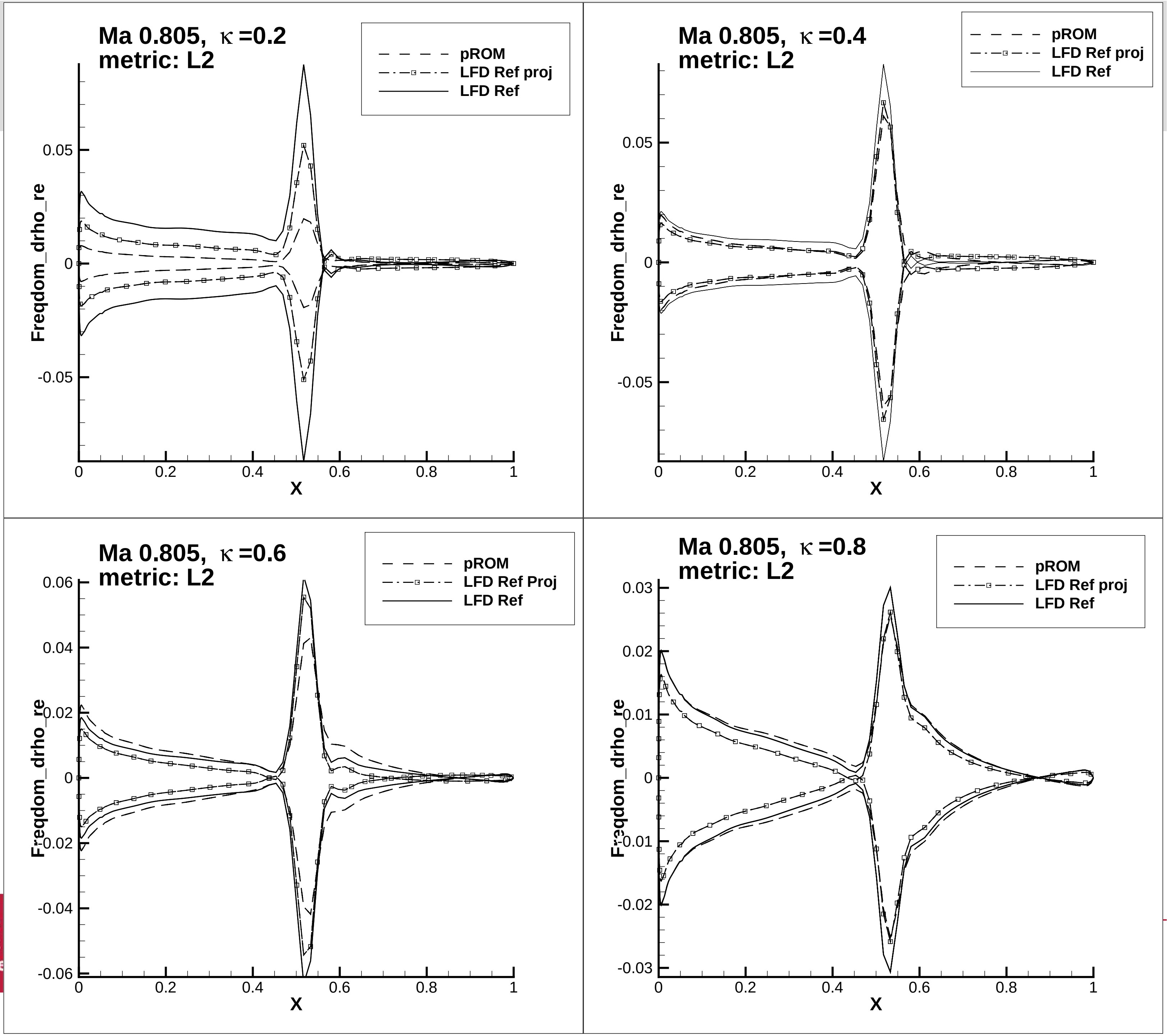
# A practical example

# Metrics matter!

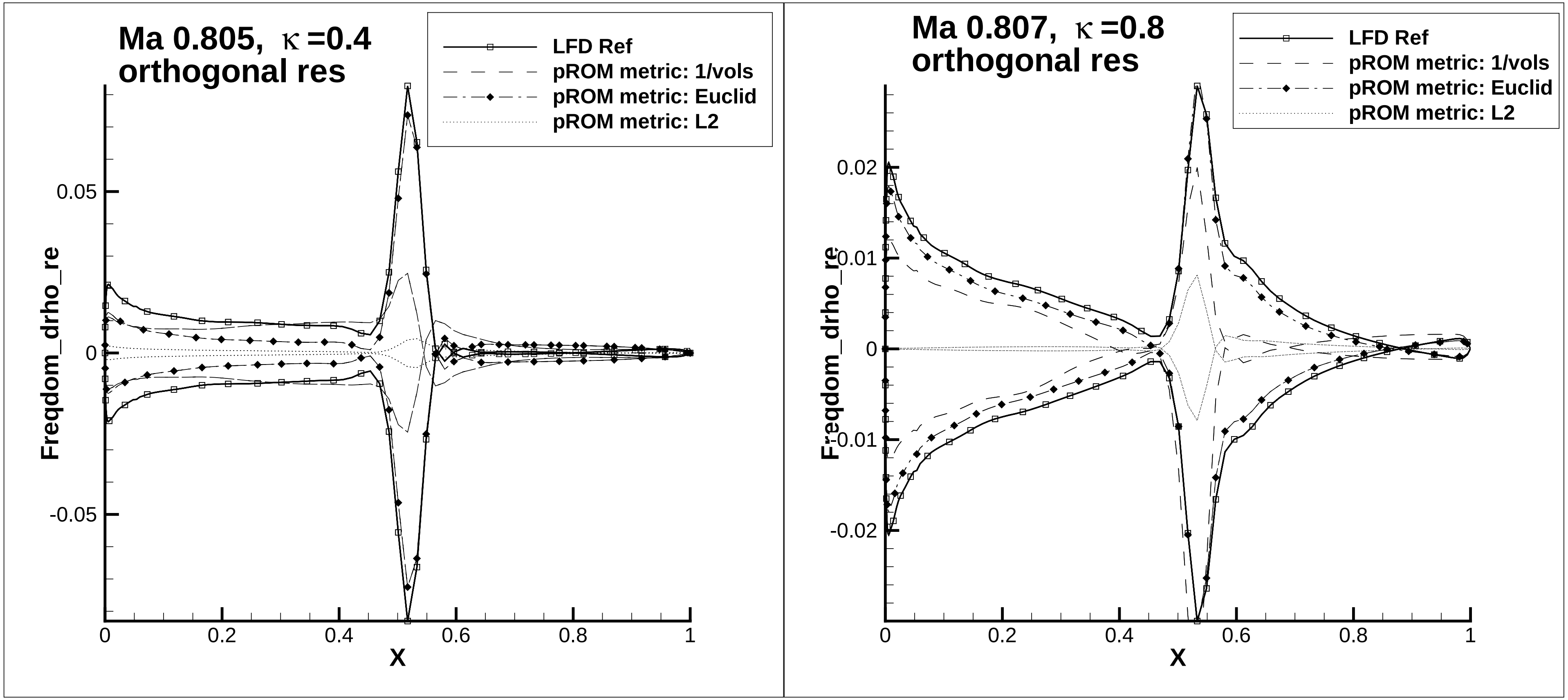
W.r.t. this metric, even the projection of the reference is quite off!

$$\langle \cdot,\cdot \rangle = (\cdot)^T \overline{M} \, (\cdot)$$



# A practical example

## The “orthogonal residual” approach is non-competitive

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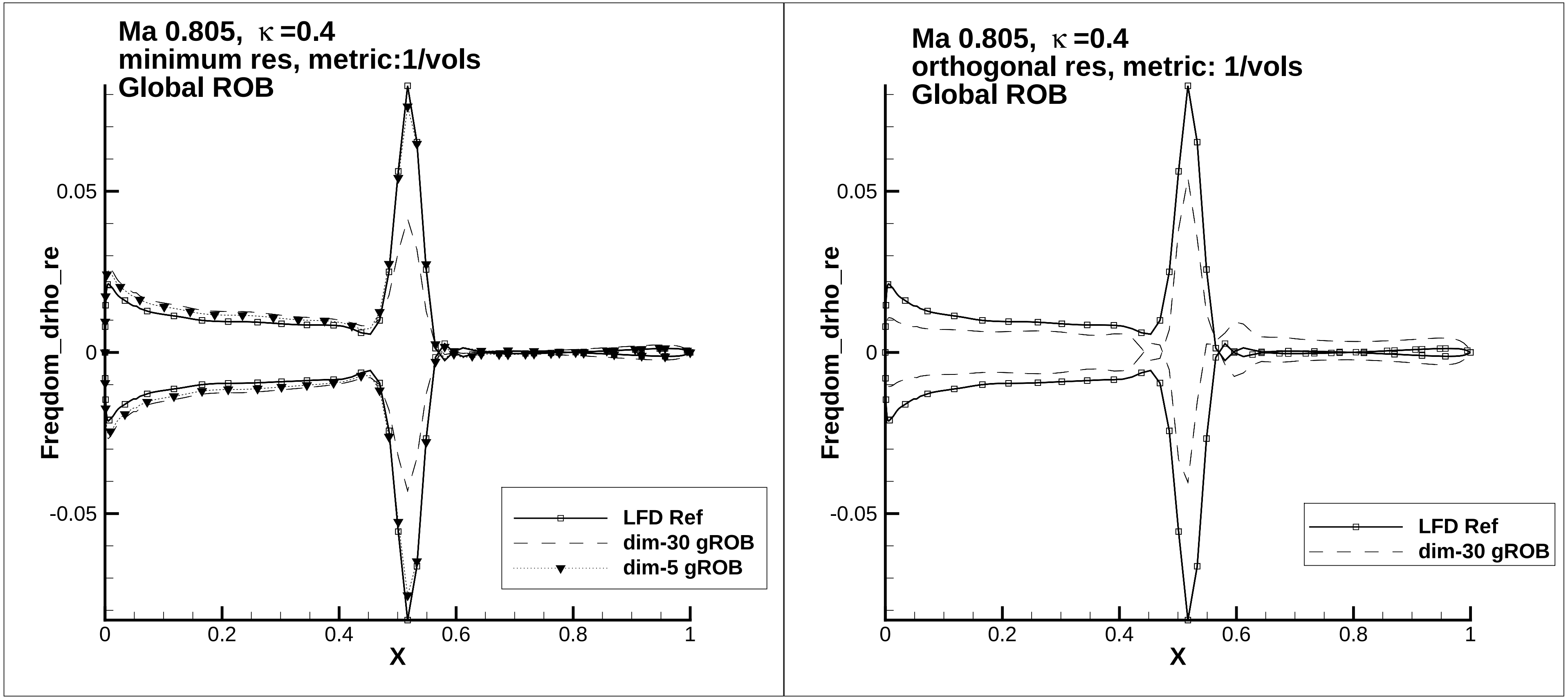
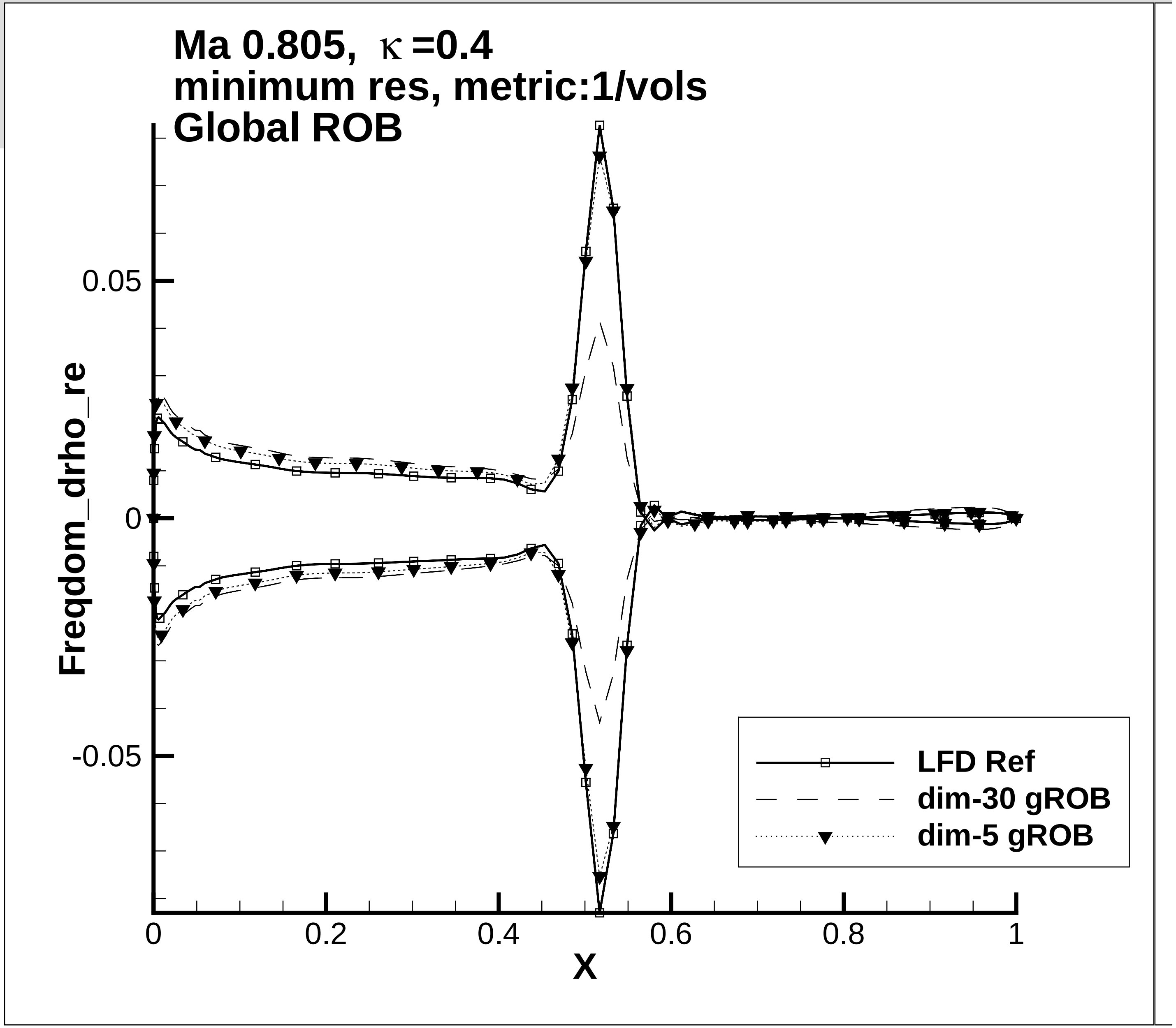
# A practical example

## The “global 2D POD basis” approach is non-competitive



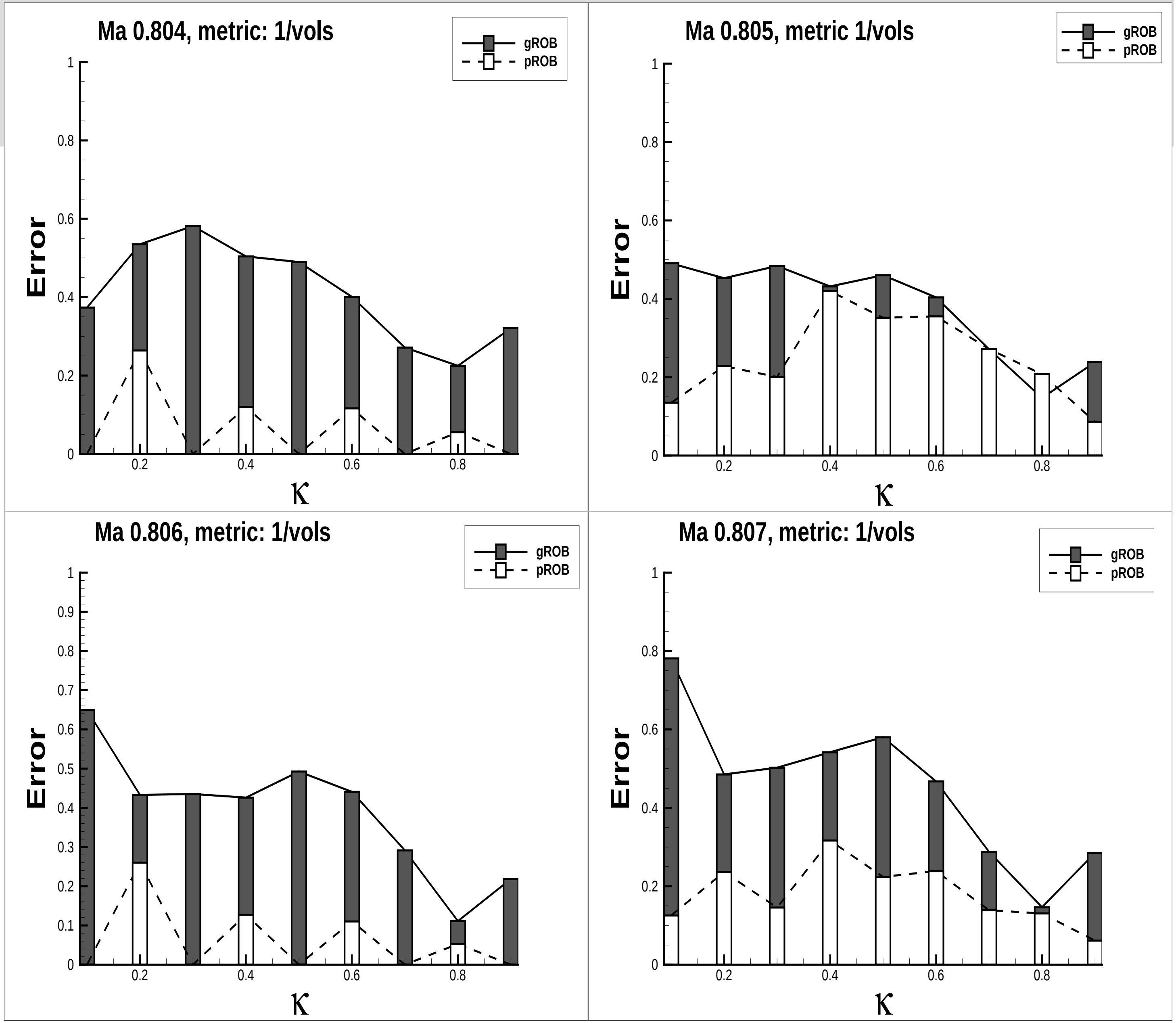

$$\langle \cdot,\cdot \rangle = (\cdot)^T \bar{M}^{-1} (\cdot)$$

**The “reduced global 2D POD basis” approach** **is competitive,** **if only the 5 most dominant POD modes are kept, but…**




$$\langle \cdot, \cdot \rangle = (\cdot)^T \bar{M}^{-1} (\cdot)$$

**…any “global basis” approach lacks the interpolation property!**



## Computational effort

- **Full order dimension:** 5*2*#(grid points) = 107,270
- **Reduced order dimension:** 5(complex) = 10 (real) at each OP

| Timing results | | | | |
|---|---|---|---|---|
| | | **Offline** | | |
| | Mean flow | Full-order LFD | Complex POD | total |
| CPU time | ~300s | 5*35s | ~0.26s | ~475s (7min) |
| | | **Semi-offline** | | |
| | Grassmann interp. | Reading system | Order reduction | |
| CPU time | ~0.66s | ~10.0s | ~0.15s | ~11s |
| | | **Online** | | |
| | Set up ROM | Solve ROM | Build flow state | |
| CPU time | Below measure acc. | Below measure | 0.01s | ~0.01s |

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**Online speed up: factor ~3500**

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- **Conclusions & Outlook**

# Conclusions & Outlook

- Parametric ROM for LFD solver introduced
- Parameters of major interest (Mach, frequency) allow for a natural division into operation points and examination parameters
- **ROM** splits into **three** stages
  - **Offline**

    (provide snapshot data, compute projection bases)
  - **Semi-Offline**

    (precompute frequency-independent system parts,

    separable parameter dependency renders an interpolation of reduced systems as in [4,5] unnecessary)
  - **Online**

    (solve reduced system, real-time performance)

[4] Degroote J. et al., 2010

[5] Panzer, H. et al., 2010

# Conclusions & Outlook

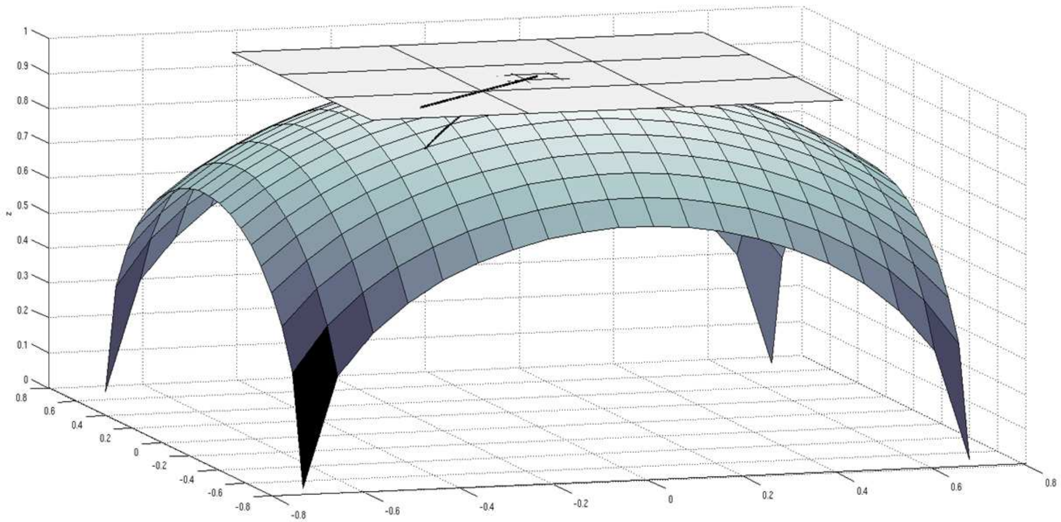
- **Metric proves to be essential in order to obtain good results**
  (requires minor technical modifications, not shown here)

- **Minimum residual approach outperforms orthogonal residual approach**

- **Reduced global basis approach gives satisfactory results, but underperforms when compared to Grassmann interpolation,**
  (Necessarily lacks interpolation property)

- **Next step: geometrically optimized reduced order basis**

# Thank you for your attention!



# An admissible interpolation scheme: radial basis functions

## Radial basis functions (RBF) in a nutshell

- Given:
  - Sample points $p^1, \dots, p^k \in R^d$
  - Sample values $y_1 = y(p^1), \dots, y_k = y(p^k) \in R$
- RBF interpolator: $y^*(p) = (y_1, \dots, y_k)\, \Psi^{-1} \psi(p)$

$$\Psi = \left(r\left(\left\| p^i - p^j \right\|\right)\right)_{i,j \le k} \in R^{k \times k}$$

$$\psi = \begin{pmatrix} r\left(\left\| p^1 - p \right\|\right) \\ \vdots \\ r\left(\left\| p^k - p \right\|\right) \end{pmatrix} \in R^k$$

# An admissible interpolation scheme: radial basis functions

## Application to Grassmann tangent vectors

- Given:
  - Sample points $p^1,\ldots,p^k \in R^d$
  - Sample values $T^1 = T(p^1),\ldots,T^k = T(p^k) \in R^{n\times m}$

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## An admissible interpolation scheme: radial basis functions

### Application to Grassmann tangent vectors

- RBF interpolator:

$$\left(T_{i,j}{}^{*}(p)\right)_{i\leq n,\, j\leq m} = \left((T_{i,j}^{1},\ldots,T_{i,j}^{k})\,\Psi^{-1}\,\psi(p)\right)_{i\leq n,\, j\leq m}$$

$$= \left((T_{:,j}^{1},\ldots,T_{:,j}^{k})\,\Psi^{-1}\,\psi(p)\right)_{j\leq m}$$

$$= \left(\sum_{l=1}^{k}\left(\Psi^{-1}\,\psi(p)\right)_{l} T_{:,1}^{l},\ldots,\sum_{l=1}^{k}\left(\Psi^{-1}\,\psi(p)\right)_{l} T_{:,m}^{l}\right)$$

$$= \sum_{l=1}^{k}\left(e_{l}^{T}\,\Psi^{-1}\,\psi(p)\right)T^{l} \qquad \boxed{= \sum_{l=1}^{k} f_{l}(p)T^{l}}$$

## An admissible interpolation scheme: radial basis functions

- **Conclusion**

  - RBF interpolation is admissible (so is e.g. linear interpolation)

  - Entry-by-entry RBF interpolation tangent matrices $\Leftrightarrow$
    interpolating coefficients of linear expansion

  - Associated computational effort:
    $nm$ interpolation procedures, each for $k$ sample points

vs.

$k$ interpolation procedures, each for $k$ sample points