Structure-Preserving Model Reduction for Nonlinear Port-Hamiltonian Systems

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Outline

1. Introduction
2. Model Reduction for Nonlinear Port-Hamiltonian
   - Structure-preserving POD and error bounds
   - Structure-preserving POD-DEIM and error bounds
   - Enriching the POD subspace
3. Numerical examples
4. Conclusions

Nonlinear Port-Hamiltonian (NPH) systems

Full-order system (dim n):

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y} &= \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{x}),\end{aligned}$$

- $\mathbf{x} \in \mathbb{R}^n$: State variable; $\mathbf{u} \in \mathbb{R}^{n_{in}}$: Input; $\mathbf{y} \in \mathbb{R}^{n_{out}}$: Output
- H: Hamiltonian - total energy in the system. $H : \mathbb{R}^n \to [0, \infty)$
- $\mathbf{J}$: Structure matrix (interconnection of energy storage components)
- $\mathbf{R}$: Dissipation matrix (describing internal energy losses)
- Structure: $\mathbf{J} = -\mathbf{J}^T$, $\mathbf{R} = \mathbf{R}^T \geq 0$. $H : \mathbb{R}^n \to [0, \infty)$
- Passive system: $H(\mathbf{x}(t_1)) - H(\mathbf{x}(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}(t)^T \mathbf{u}(t) dt.$
- Generalizes classical Hamiltonian systems: $\dot{\mathbf{x}} = \mathbf{J}\nabla_{\mathbf{x}} H(\mathbf{x})$.
- [van der Schaft, 2006], [Zwart/Jacob, 2009]
- **Applications:** Circuit, Network/interconnect structure, Mechanics (Euler-Lagrange eqn), e.g. Toda Lattice, Ladder Network

Model Reduction

Full-order system (dim n):

$$\begin{aligned}\dot{\mathbf{x}} &= (\mathbf{J}-\mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t)\\ \mathbf{y} &= \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{x}),\end{aligned}$$

GOAL: Reduced system (dim $r \ll n$):

$$\begin{aligned}\dot{\mathbf{x}}_r &= (\mathbf{J}_r-\mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r\mathbf{u}(t)\\ \mathbf{y}_r &= \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r),\end{aligned}$$

- $\mathbf{J} = -\mathbf{J}^T$, $\mathbf{R} = \mathbf{R}^T \geq 0$. Hamiltonian: $H : \mathbb{R}^n \to [0,\infty)$, $H(\mathbf{x}) > 0$, $H(\mathbf{0}) = 0$

  " Preserve Structure, Stability & Passivity"
- $\mathbf{J}_r = -\mathbf{J}_r^T$, $\mathbf{R}_r = \mathbf{R}_r^T \geq 0$. Hamiltonian: $H_r : \mathbb{R}^r \to [0,\infty)$, $H_r(\mathbf{x}_r) > 0$, $H_r(\mathbf{0}) = 0$
- $$H_r(\mathbf{x}_r(t_1)) - H_r(\mathbf{x}_r(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}_r(t)^T \mathbf{u}(t)dt.$$

Model Reduction via Petrov-Galerkin Projection

Choose basis matrices $\mathbf{V}_r \in \mathbb{R}^{n \times r}$ and $\mathbf{W}_r \in \mathbb{R}^{n \times r}$ so that

- $\mathbf{x} \approx \mathbf{V}_r \mathbf{x}_r$ ($\mathbf{x}(t)$ approximately lives in an r-dimensional subspace)
- $\mathrm{Span}\{\mathbf{W}_\mathrm{r}\}$ is orthogonal to the residual:

$$\mathbf{W}_r^T \quad [\mathbf{V}_r \dot{\mathbf{x}}_r(t) - (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r) - \mathbf{B}\mathbf{u}(t)] = \mathbf{0}$$
$$\mathbf{y}_r(t) = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r).$$

- and with $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$ (change of basis)

$$\dot{\mathbf{x}}_r = \mathbf{W}_r^T (\mathbf{J} - \mathbf{R}) \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r) + \mathbf{W}_r^T \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}_r = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r),$$

Two Main Issues:

- Port-Hamiltonian structure is not preserved $\Longrightarrow$ Stability and passivity of the reduced model are not guaranteed.
- The complexity is not truly reduced – complexity of nonlinear term $\sim \mathcal{O}(n)$

Model Reduction for Nonlinear Port-Hamiltonian [Beattie & G. (2011)] [1]

- [Fujimoto, H. Kajiura (2007)], [Scherpen, van der Schaft (2008)]
- Find $\mathbf{V}_r$ such that $\mathbf{x}(t) \approx \mathbf{V}_r \mathbf{x}_r(t)$
- Find $\mathbf{W}_r$ such that $\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \mathbf{W}_r\, \mathbf{c}(t) \quad \text{for} \quad \text{some} \quad \mathbf{c}(\mathrm{t}) \in \mathbb{R}^{\mathrm{r}}$

$$\nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \;\approx\; \nabla_{\mathbf{x}} H(\mathbf{x}(t)) \;\approx\; \mathbf{W}_r\, \mathbf{c}(t)$$

- $\mathbf{V}_r^T \mathbf{W}_r = \mathbf{I}$,

$$\Longrightarrow \mathbf{c}(t) = \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \quad = \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$$

Reduced-order Hamiltonian:

$$H_r(\mathbf{x}_r(t)) := H(\mathbf{V}_r \mathbf{x}_r(t))$$

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$$\nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \;\approx\; \nabla_{\mathbf{x}} H(\mathbf{x}(t)) \;\approx\; \mathbf{W}_r\, \mathbf{c}(t)$$

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Reduced-order Hamiltonian:

$$H_r(\mathbf{x}_r(t)) := H(\mathbf{V}_r \mathbf{x}_r(t))$$

- Substitute $\mathbf{x} \longrightarrow \mathbf{V}_r \mathbf{x}_r$, and $\nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r(t)) \longrightarrow \mathbf{W}_r \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r(t))$ with

$$\mathbf{W}_r{}^T \left[\mathbf{V}_r \dot{\mathbf{x}}_r - (\mathbf{J} - \mathbf{R}) \mathbf{W}_r \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r) + \mathbf{B}\mathbf{u}(t) = 0 \right], \qquad \mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}.$$

Reduced system:

$$\dot{\mathbf{x}}_r \quad = \quad (\mathbf{J}_r - \mathbf{R}_r) \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r \mathbf{u}(t), \qquad \mathbf{y}_r \;=\; \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r),$$

where $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r$, $\mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$, $\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) = \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r)$.

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Reduced system (dim $r \ll n$):

$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r\mathbf{u}(t) \qquad \mathbf{y}_r = \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r),$$

- $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r, \quad \mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r, \quad \mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}, \quad \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) = \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r).$
- Preserve Structure & Passivity :

  $\mathbf{J}_r = -\mathbf{J}_r^T, \mathbf{R}_r = \mathbf{R}_r^T \geq 0. \quad H_r : \mathbb{R}^r \to [0, \infty), H_r := H(\mathbf{V}_r \mathbf{x}_r)$

  $H_r(\mathbf{x}_r(t_1)) - H_r(\mathbf{x}_r(t_0)) \leq \int_{t_0}^{t_1} \mathbf{y}_r(t)^T \mathbf{u}(t) dt.$

Choices of Basis matrices $\mathbf{V}_r$ and $\mathbf{W}_r$:

- **Proper Orthogonal Decomposition (POD)**
- Quasi-optimal $\mathcal{H}_2$ basis.

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Proper Orthogonal Decomposition (POD):

- Extracts (orthonormal) basis containing dominant characteristics of the system

- Optimal with respect to **Snapshot**: $\mathbb{X} := \{\mathbf{x}(t) | t \in [0, T]\}$
- POD basis $\mathbf{V}_r = [\mathbf{v}_1, \ldots, \mathbf{v}_r]$:, $\leftrightsquigarrow$ $\boxed{\mathrm{Span}\{\mathbf{V}_\mathrm{r}\} \approx \mathrm{Span}\{\mathbf{x}(\mathrm{t}) | \mathrm{t} \in [0, \mathrm{T}]\}}$

POD =Optimal Solution:

$$\min_{\mathrm{rank}\{\mathbf{V}_r\}=r} \int_0^T \|\mathbf{x}(t) - \mathbf{V}_r \mathbf{V}_r^T \mathbf{x}(t)\|^2 \, dt, \quad \text{s.t.} \quad \langle \mathbf{v}_i, \mathbf{v}_j \rangle = \delta_{ij}$$

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- Extracts (orthonormal) basis containing dominant characteristics of the system
- Optimal with respect to **Snapshot**: $\mathbb{X} := \{\mathbf{x}(t) | t \in [0, T]\}$
- POD basis $\mathbf{V}_r = [\mathbf{v}_1, \dots, \mathbf{v}_r]$:, $\rightsquigarrow$ $\boxed{\mathrm{Span}\{\mathbf{V}_r\} \approx \mathrm{Span}\{\mathbf{x}(t) | t \in [0, T]\}}$

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- **Nonlinear Snapshot**: $\mathbb{F} := \{\mathbf{F}(t) | t \in [0, T]\}$, $\mathbf{F}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t))$.
- POD Basis: $\mathbf{W}_r \in \mathbb{R}^{n \times r}$, $\rightsquigarrow$ $\boxed{\mathrm{Span}\{\mathbf{W}_r\} \approx \mathrm{Span}\{\nabla_{\mathbf{x}} \mathrm{H}(\mathbf{x}(t)) | t \in [0, T]\}}$

Error

$$\int_0^T \|\mathbf{x}(t) - \mathbf{V}_r\mathbf{V}_r^T\mathbf{x}_j\|_2^2 \, dt \;=\; \sum_{\ell=r+1}^{n_t} \lambda_\ell, \qquad \int_0^T \|\mathbf{F}(t) - \mathbf{W}_r\mathbf{W}_r^T\mathbf{F}(t)\|_2^2 \, dt \;=\; \sum_{\ell=r+1}^{n_t} \varrho_\ell.$$

- Eigen-Decomp: $\int_0^T \mathbf{x}(t)\mathbf{x}(t)^T dt = \widehat{\mathbf{V}}\Lambda\widehat{\mathbf{V}}^T$, $\mathbf{V}_r = \widehat{\mathbf{V}}(:, 1:r)$,
  $\int_0^T \mathbf{F}(t)\mathbf{F}(t)^T dt = \widehat{\mathbf{W}}\mathbf{D}\widehat{\mathbf{W}}^T$, $\mathbf{W}_r = \widehat{\mathbf{W}}(:, 1:r)$.
- In practice, use *SVD* of discrete snapshots e.g. $\mathbb{X} := \{\mathbf{x}_1, \dots, \mathbf{x}_{n_t}\}$, $\mathbf{x}_j := \mathbf{x}(t_j)$.

POD for port-Hamiltonian systems (POD-PH)

Algorithm (POD-based MOR for port-Hamiltonian systems [Beattie, G. (2011)])

1. *Generate trajectory* $\mathbf{x}(t)$*, and collect snapshots:*
$$\mathbb{X} = [\mathbf{x}(t_0), \mathbf{x}(t_1), \mathbf{x}(t_2), \ldots, \mathbf{x}(t_N)]\,.$$
2. *Truncate SVD of snapshot matrix,* $\mathbb{X}$*, to get POD basis,* $\widetilde{\mathbf{V}}_r$*, for the state space variables. Then*
$$\mathbf{x}(t) \approx \widetilde{\mathbf{V}}_r \tilde{\mathbf{x}}_r(t)$$
3. *Collect associated force snapshots:*
$$\mathbb{F} = [\nabla_{\mathbf{x}} H(\mathbf{x}(t_0)), \nabla_{\mathbf{x}} H(\mathbf{x}(t_1)), \ldots, \nabla_{\mathbf{x}} H(\mathbf{x}(t_N))]\,.$$
4. *Truncate SVD of* $\mathbb{F}$ *to get a second POD basis,* $\widetilde{\mathbf{W}}_r$*, spanning approximate range of*
$$\nabla_{\mathbf{x}} H(\mathbf{x}(t)) \approx \widetilde{\mathbf{W}}_r \tilde{\mathbf{f}}_r(t).$$
5. *Change bases* $\widetilde{\mathbf{W}}_r \mapsto \mathbf{W}_r$ *and* $\widetilde{\mathbf{V}}_r \mapsto \mathbf{V}_r$ *such that* $\mathbf{W}_r^T \mathbf{V}_r = \mathbf{I}$.

The POD-PH reduced system is
$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r \mathbf{u}(t), \qquad \mathbf{y}_r(t) = \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r)$$
with $\mathbf{J}_r = \mathbf{W}_r^T \mathbf{J} \mathbf{W}_r$, $\mathbf{R}_r = \mathbf{W}_r^T \mathbf{R} \mathbf{W}_r$, $\mathbf{B}_r = \mathbf{W}_r^T \mathbf{B}$, and $H_r(\mathbf{x}_r) = H(\mathbf{V}_r \mathbf{x}_r)$.

A-Priori Error for NPH from structure preserving MOR

Error bounds for State variable & Output [Chaturantaut, Beattie & Gugercin (2013)]:

Basis matrices $\mathbf{V}_r, \mathbf{W}_r$ with $\mathbf{W}_r^T\mathbf{V}_r = \mathbf{V}_r^T\mathbf{W} = \mathbf{I}$ and $\mathbf{V}_r^T\mathbf{V}_r = \mathbf{I}$,

$$\int_0^T \|\mathbf{x}(t) - \mathbf{V}_r\mathbf{x}_r(t)\|^2 dt \quad \leq \quad C_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + C_{\mathbf{f}} \sum_{\ell=r+1}^{n_t} \varrho_\ell$$

and

$$\int_0^T \|\mathbf{y}(t) - \mathbf{y}_r(t)\|^2 dt \quad \leq \quad \widehat{C}_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + \widehat{C}_F \sum_{\ell=r+1}^{n_t} \varrho_\ell$$

$\Longrightarrow$ *Error bounds are proportional to the least-squares errors ($\mathcal{L}_2$-norm) of snapshots* $\mathbf{x}(t)$ *and* $\mathbf{F}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t))$.

- $\widehat{C}_{\mathbf{x}} = 2\widehat{\beta} C_{\mathbf{x}}$, $\widehat{C}_F = 2\|\mathbf{B}^T(\mathbf{I} - \mathbf{W}_r\mathbf{V}_r^T)\|^2 + 2\widehat{\beta} C_{\mathbf{f}}$, $\widehat{\beta} = \|\mathbf{B}^T\mathbf{W}_r\mathbf{V}_r^T\|^2 L_f^2$
- $C_{\mathbf{x}} = 2(\|\mathbf{I} - \mathbf{V}_r\mathbf{W}_r^T\|^2 + 2Tc_\alpha(T)\beta^2)$, $\quad C_{\mathbf{f}} = 4Tc_\alpha(T)\gamma^2$
  $$c_\alpha(T) = \begin{cases} \frac{1}{\alpha}\left[e^{2\alpha T} - 1\right] & , \quad \alpha \neq 0 \\ 2T & , \quad \alpha = 0 \end{cases}$$
- $\alpha = M[\mathbf{A}_r\mathbf{F}_r]$, $\beta = \|\mathbf{W}_r^T(\mathbf{J} - \mathbf{R})\mathbf{W}_r\mathbf{V}_r^T\| L_f \|(\mathbf{I} - \mathbf{V}_r\mathbf{W}_r^T)\|$, , $\gamma = \|\mathbf{W}_r^T(\mathbf{J} - \mathbf{R})(\mathbf{I} - \mathbf{W}_r\mathbf{V}_r^T)\|$.

DEIM with Structure Preserving Model Reduction

Recall reduced-order NPH:

$$\dot{\mathbf{x}}_r = (\mathbf{J}_r - \mathbf{R}_r)\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) + \mathbf{B}_r \mathbf{u}(t) \qquad \mathbf{y}_r = \mathbf{B}_r^T \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r),$$

where $\nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) = \mathbf{V}_r^T \nabla_{\mathbf{x}} H(\mathbf{V}_r \mathbf{x}_r)$.

- Let $\mathbf{F}(x) = \nabla_{\mathbf{x}} H(\mathbf{x})$ and $\mathbf{F}_r(\mathbf{x}_r) := \nabla_{\mathbf{x}_r} H_r(\mathbf{x}_r) = \mathbf{V}_r^T \mathbf{F}(\mathbf{V}_r \mathbf{x}_r)$
- $\Rightarrow$ Complexity of nonlinear term $\sim \mathcal{O}(n)$
- Goals: (i) Low complexity with accuracy maintained (ii) Structure preserved

[3] A. Hochman, B. N. Bond, and J. K. White. *A Stabilized Discrete Empirical Interpolation Method for Model Reduction of Electrical, Thermal, and Microelectromechanical Systems*. the 48th Design Automation Conference (DAC), 2011

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(i) ▷ For accuracy, re-write[3] : $\mathbf{F}(\mathbf{x}) = \mathbf{Q}\mathbf{x} + \mathbf{f}(\mathbf{x}), \qquad \mathbf{f}(\mathbf{x}) := \mathbf{F}(\mathbf{x}) - \mathbf{Q}\mathbf{x}$

▷ Complexity reduction: $\mathbf{F}_r(\mathbf{x}_r) := \underbrace{\mathbf{V}_r^T \mathbf{Q} \mathbf{V}_r}_{\text{Precomp.: } r \times r} \mathbf{x}_r + \underbrace{\mathbf{f}_r(\mathbf{x}_r)}_{\text{Use "DEIM"}}$

$\mathbf{f}_r(\mathbf{x}_r) = \mathbf{V}_r^T \mathbf{f}(\mathbf{V}_r \mathbf{x}_r) \Rightarrow$ Complexity$\sim \mathcal{O}(n)$ ▷ **Discrete Empirical Interpolation (DEIM)**

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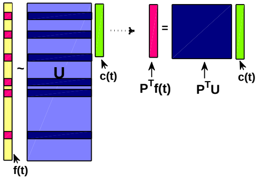
Discrete Empirical Interpolation Method (DEIM) [4]

- $\mathbf{f}(t) := \mathbf{f}(\mathbf{x}(t))$
- POD basis $\mathbf{U} \in \mathbb{R}^{n\times m}$ of $\mathbf{f}(t)$, $m \ll n$
- $\mathbf{f}(t) \approx \mathbf{U}\mathbf{c}(t)$

DEIM: $\mathbf{f}(t) \approx \hat{\mathbf{f}}(t) := \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T\mathbf{f}(t),$



- $\mathbf{P}^T$ *"extracts m rows"* $\wp_1, \ldots, \wp_m$.
- $\wp := [\wp_1, \ldots, \wp_m]$
- e.g. $\mathbf{P}^T\mathbf{U} = \mathbf{U}(\wp, :)$
- $\mathbf{P} = [\mathbf{e}_{\wp_1}, \ldots, \mathbf{e}_{\wp_m}]$, $\mathbf{e}_{\wp_i} = \wp_i$-th column of $\mathbf{I}_n$

$$\mathbf{f}_r(\mathbf{x}_r) = \underbrace{\mathbf{V}_r^T}_{r\times n} \underbrace{\mathbf{f}(\mathbf{V}_r\mathbf{x}_r(t))}_{n\times 1}$$

$$\approx \underbrace{\mathbf{V}_r^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}}_{\text{precomp }: r\times m} \underbrace{\mathbf{P}^T\mathbf{f}(\mathbf{V}_r\mathbf{x}_r)}_{m\times 1} := \hat{\mathbf{f}}_r(\mathbf{x}_r)$$

$\Longrightarrow$ Reduced Complexity: from n to m

[4] S. Chaturantabut and D. C. Sorensen. *Discrete empirical interpolation for nonlinear model reduction.* SIAM J. Sci. Comput., 32(5): pp. 2737-2764, 2010.

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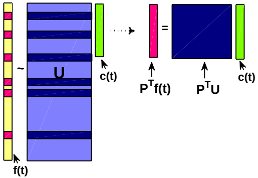

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$$\mathbf{f}_r(\mathbf{x}_r) = \underbrace{\mathbf{V}_r^T}_{r\times n} \underbrace{\mathbf{f}(\mathbf{V}_r\mathbf{x}_r(t))}_{n\times 1}$$

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$\Longrightarrow$ Reduced Complexity: from n to m

▷ **DEIM** : $\hat{\mathbf{f}}_r(\mathbf{x}_r) := \mathbf{V}_r^T\mathbb{P}\,\mathbf{f}(\mathbf{V}_r\mathbf{x}_r) \Longrightarrow$ *Structure is not preserved!*

where $\mathbb{P} = \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T$

(ii) **DEIM with structure preserved**- Want: $\mathbf{f}_r(\mathbf{x}_r) \longleftarrow \mathbf{Z}^T\mathbf{f}(\mathbf{Z}\mathbf{x}_r)$ $\mathbf{Z} : n \times r$

Existing approach: see e.g. [Carlberg *et al.* , 2012]

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DEIM with Structure Preserving Model Reduction

DEIM : $\hat{\mathbf{f}}_r(\mathbf{x}_r) := \mathbf{V}_r^T \mathbb{P}\, \mathbf{f}(\mathbf{V}_r\mathbf{x}_r) = \mathbf{V}_r^T \mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{P}^T\mathbf{f}(\mathbf{V}_r\mathbf{x}_r)$.

To preserve structure, use 2 approximations:

$$(1)\quad \mathbf{P}^T\mathbf{f}(\mathbf{V}_r\mathbf{x}_r) \approx \mathbf{P}^T\mathbf{f}(\mathbf{P}\mathbf{P}^T\mathbf{V}_r\mathbf{x}_r) \qquad \longleftarrow \text{Exact for component-wise } \mathbf{f}$$

$$(2)\quad \mathbf{P}^T\mathbf{f}(\mathbf{P}\mathbf{P}^T\mathbf{V}_r\mathbf{x}_r) \approx \mathbf{P}^T\mathbf{V}_r\widehat{\mathbf{c}} \qquad \Longrightarrow \widehat{\mathbf{c}} = \mathit{arg}\min_{\mathbf{c}\in\mathbb{R}^m} \|\mathbf{P}^T\mathbf{f}(\mathbf{P}\mathbf{P}^T\mathbf{V}_r\mathbf{x}_r) - \mathbf{P}^T\mathbf{V}_r\mathbf{c}\|_2$$

$$\Longrightarrow \qquad \mathbf{P}^T\mathbf{f}(\mathbf{V}_r\mathbf{x}_r) \approx \mathbf{P}^T\mathbb{Q}\,\mathbf{f}(\mathbf{V}_p\mathbf{x}_r), \qquad \text{or} \qquad \mathbf{P}^T\mathbf{f}(\mathbf{V}_r\mathbf{x}_r) \approx \mathbf{M}\mathbf{V}_p^T\mathbf{f}(\mathbf{V}_p\mathbf{x}_r)$$

Projector $\mathbb{Q} := \mathbf{V}_r\left[(\mathbf{P}^T\mathbf{V}_r)^T(\mathbf{P}^T\mathbf{V}_r)\right]^{-1}(\mathbf{P}^T\mathbf{V}_r)^T\mathbf{P}^T$,

$\mathbf{M} = \mathbf{P}^T\mathbf{V}_r\left[(\mathbf{P}^T\mathbf{V}_r)^T(\mathbf{P}^T\mathbf{V}_r)\right]^{-1}, \quad \mathbf{V}_p = \mathbf{P}\mathbf{P}^T\mathbf{V}_r.$

- $\widetilde{\mathbf{W}}_r \longleftarrow \mathbf{W}_r\mathbf{V}_r^T\mathbf{U}(\mathbf{P}^T\mathbf{U})^{-1}\mathbf{M}$

"Standard DEIM" $\Longrightarrow$ *"Structure-Preserving DEIM"*

$$\underbrace{\mathbf{W}_r^T(\mathbf{J}-\mathbf{R})\mathbf{W}_r}_{\surd}\,\underbrace{\mathbf{V}_r^T\mathbb{P}\,\mathbf{f}(\mathbf{V}_r\mathbf{x}_r)}_{\times} \Longrightarrow \underbrace{\widetilde{\mathbf{W}}_r^T(\mathbf{J}-\mathbf{R})\widetilde{\mathbf{W}}_r}_{\surd}\,\underbrace{\mathbf{V}_p^T\mathbf{f}(\mathbf{V}_p\mathbf{x}_r)}_{\surd}$$

DEIM with Structure Preserving Reduced System

$$\dot{\mathbf{x}}_r = (\widetilde{\mathbf{J}}_r - \widetilde{\mathbf{R}}_r)\nabla_{\mathbf{x}_r}\widetilde{H}_r(\mathbf{x}_r) + \widetilde{\mathbf{B}}_r\mathbf{u}(t) \qquad \mathbf{y}_r = \widetilde{\mathbf{B}}_r^T\nabla_{\mathbf{x}_r}\widetilde{H}_r(\mathbf{x}_r).$$

- $\widetilde{\mathbf{J}}_r = \widetilde{\mathbf{W}}_r^T\mathbf{J}\widetilde{\mathbf{W}}_r$, $\widetilde{\mathbf{R}}_r = \widetilde{\mathbf{W}}_r^T\mathbf{R}\widetilde{\mathbf{W}}_r$, $\widetilde{\mathbf{B}}_r = \widetilde{\mathbf{W}}_r^T\mathbf{B}$, $\nabla_{\mathbf{x}_r}\widetilde{H}_r(\mathbf{x}_r) := \mathbf{Q}_r\mathbf{x}_r + \mathbf{V}_p^T\mathbf{f}(\mathbf{V}_p\mathbf{x}_r)$

Reduced-order Hamiltonian:

$$\widetilde{H}_r(\mathbf{x}_r) := H(\mathbf{V}_p\mathbf{x}_r) + \frac{1}{2}\mathbf{x}_r^T\mathbf{V}_r^T\left[\mathbf{Q} - \mathbf{P}\mathbf{P}^T\mathbf{Q}\mathbf{P}\mathbf{P}^T\right]\mathbf{V}_r\mathbf{x}_r$$

$$\widetilde{H}_r(\mathbf{x}_r(t_1)) - \widetilde{H}_r(\mathbf{x}_r(t_0)) \leq \int_{t_0}^{t_1}\mathbf{y}_r(t)^T\mathbf{u}(t)dt.$$

$\widetilde{\mathbf{J}}_r = -\widetilde{\mathbf{J}}_r^T$, $\widetilde{\mathbf{R}}_r = \widetilde{\mathbf{R}}_r^T \geq 0$, $\exists\widetilde{H}_r \implies$ **Preserve Structure and Passivity**

A-Priori Error for NPH from DEIM-structure preserving MOR

Error bounds for State variable & Output:

Using the notations defined earlier:

$$\int_0^T \|\mathbf{x}(t) - \mathbf{V}_r \mathbf{x}_r(t)\|^2 dt \quad \leq \quad C_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + C_{\mathbf{F}} \sum_{\ell=r+1}^{n_t} \varrho_\ell + C_{\mathbf{f}} \sum_{\ell=m+1}^{n_t} s_\ell + C_0$$

and

$$\int_0^T \|\mathbf{y}(t) - \mathbf{y}_r(t)\|^2 dt \quad \leq \quad \widehat{C}_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + \widehat{C}_{\mathbf{F}} \sum_{\ell=r+1}^{n_t} \varrho_\ell + \widehat{C}_{\mathbf{f}} \sum_{\ell=m+1}^{n_t} s_\ell + \widehat{C}_0$$

$\Longrightarrow$ *Error bounds are proportional to the least-squares errors ($\mathcal{L}_2$-norm) of snapshots:* $\mathbf{x}(t)$, $\mathbf{F}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t))$, $\mathbf{f}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t)) - \mathbf{Q}\mathbf{x}(t)$.

A-Priori Error for NPH from DEIM-structure preserving MOR

Error bounds for State variable & Output:

Using the notations defined earlier:

$$\int_0^T \|\mathbf{x}(t) - \mathbf{V}_r \mathbf{x}_r(t)\|^2 dt \quad \leq \quad C_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + C_{\mathbf{F}} \sum_{\ell=r+1}^{n_t} \varrho_\ell + C_{\mathbf{f}} \sum_{\ell=m+1}^{n_t} s_\ell + C_0$$

and

$$\int_0^T \|\mathbf{y}(t) - \mathbf{y}_r(t)\|^2 dt \quad \leq \quad \widehat{C}_{\mathbf{x}} \sum_{\ell=r+1}^{n_t} \lambda_\ell + \widehat{C}_{\mathbf{F}} \sum_{\ell=r+1}^{n_t} \varrho_\ell + \widehat{C}_{\mathbf{f}} \sum_{\ell=m+1}^{n_t} s_\ell + \widehat{C}_0$$

$\Longrightarrow$ *Error bounds are proportional to the least-squares errors ($\mathcal{L}_2$-norm) of snapshots:* $\mathbf{x}(t)$, $\mathbf{F}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t))$, $\mathbf{f}(t) = \nabla_{\mathbf{x}} H(\mathbf{x}(t)) - \mathbf{Q}\mathbf{x}(t)$.

- $c_\alpha(T) = \begin{cases} \frac{1}{\alpha}\left[e^{2\alpha T} - 1\right] & , \quad \alpha \neq 0 \\ 2T & , \quad \alpha = 0 \end{cases}$, $\alpha = M[(\mathbf{J}_r - \mathbf{R}_r)\widetilde{\mathbf{F}}_r]$

  $c_T = 8Tc_\alpha(T)$, $\gamma = \|\mathbf{I} - \mathbf{V}\mathbf{W}^T\|$, $\beta = L_F\gamma$, $a = \|\mathbf{J} - \mathbf{R}\|$, $d_T = \|\mathbf{B}\|^2(c_T\widehat{L}^2 a^2 + 1)$
- $C_{\mathbf{x}} = 2\gamma^2 a^2 + c_T\beta^2$, $\quad C_{\mathbf{F}} = c_T\gamma^2 a^2$, $\quad C_{\mathbf{f}} = c_T a^2$, $\quad C_0 = c_T K^2$
- $\widehat{C}_{\mathbf{x}} = d_T\beta^2$, $\quad \widehat{C}_{\mathbf{F}} = d_T\gamma^2$, $\quad \widehat{C}_{\mathbf{f}} = d_T$, $\quad \widehat{C}_0 = c_T K^2 + 8\widetilde{K}^2 T$, $\quad K, \widetilde{K} \to 0$ as $m \to n$

$K, \widetilde{K} \to 0$ as $m \to n$

An Alternate Approach

- POD provides one set of choices for $\mathbf{V}_r$ and $\mathbf{W}_r$. Others may be considered:
- Find a choice of subspaces that is *asymptotically optimal* for small $\mathbf{u}$ (hence for small $\mathbf{x}$).
- $\nabla_{\mathbf{x}} H(\mathbf{x}) \approx \mathbf{Q}\mathbf{x}$ for a symmetric positive semidefinite $\mathbf{Q} \in \mathbb{R}^{n \times n}$.
- Leads to consideration of *Linear Port-Hamiltonian Systems*

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{J} - \mathbf{R})\mathbf{Q}\mathbf{x} + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{B}^T\mathbf{Q}\mathbf{x} \end{aligned} \quad \longrightarrow \quad \begin{aligned} \dot{\mathbf{x}}_r &= (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}(t) \\ \mathbf{y}_r(t) &= \mathbf{B}_r^T\mathbf{Q}_r\mathbf{x}_r \end{aligned}$$

(Original system) (Reduced system)

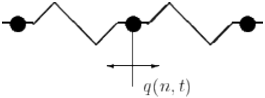
- $\mathcal{G}(s) = \mathbf{B}^T\mathbf{Q}(s\mathbf{I} - (\mathbf{J} - \mathbf{R})\mathbf{Q})^{-1}\mathbf{B} \quad \longrightarrow \quad \mathcal{G}_r(s) = \mathbf{B}_r^T\mathbf{Q}_r(s\mathbf{I} - (\mathbf{J}_r - \mathbf{R}_r)\mathbf{Q}_r)^{-1}\mathbf{B}_r$
- Find $\mathbf{V}_r$ and $\mathbf{W}_r$ that are optimal reduction spaces for $\|\mathcal{G} - \mathcal{G}_r\|_{\mathcal{H}_2}$, use them to reduce the original nonlinear system
- We use Quasi-$\mathcal{H}_2$ optimal subspaces using PH-IRKA method of [G./Polyuga/Beatie/van der Schaft/09]
- One can also use optimal bilinear spaces from [Benner, Breiten (2011)]

Numerical Test I: Toda Lattice

- 1-D motion of N-particle chain with nearest neighbor exponential interactions.
- E.g. crystal model in solid state physics.

$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{y} = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{x}).$$

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{bmatrix} \mathbf{R} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{diag}(\gamma_1, \ldots, \gamma_N) \end{bmatrix} \in \mathbb{R}^{n \times n}, \mathbf{B} = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$



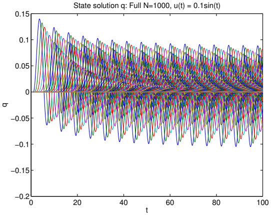
- State variable: $\mathbf{x} = \begin{bmatrix} \mathbf{q} \\ \mathbf{p} \end{bmatrix}$; q_j =displacement; p_j =momentum.
- Hamiltonian:

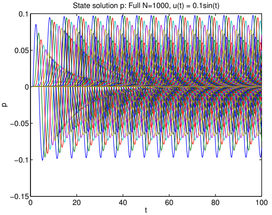
$$H = \sum_{k=1}^{N} \frac{1}{2} p_k^2 + \sum_{k=1}^{N-1} \exp(q_k - q_{k+1}) + \exp(q_N) - q_1 .$$

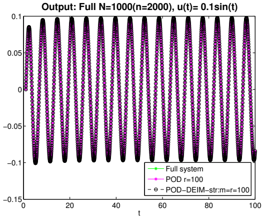
- $\mathbf{Q} := \nabla^2 \mathbf{H}(0)$, $N = 1000$; Full dim $n = 2N = 2000$.
- $\gamma_j = 0.1, j = 1, \ldots, N$

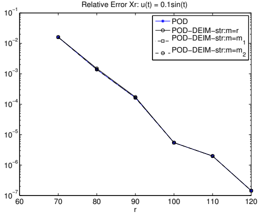
Input: $\mathbf{u}(t) = 0.1\sin(t)$

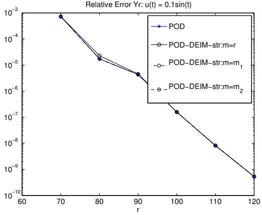
- POD basis dimension r
- DEIM dim.: $m = r, m_1, m_2, \quad m_1 = r + \texttt{ceil}(r/3), \quad m_2 = r + \texttt{ceil}(2r/3)$.

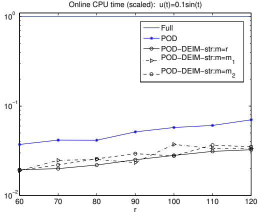






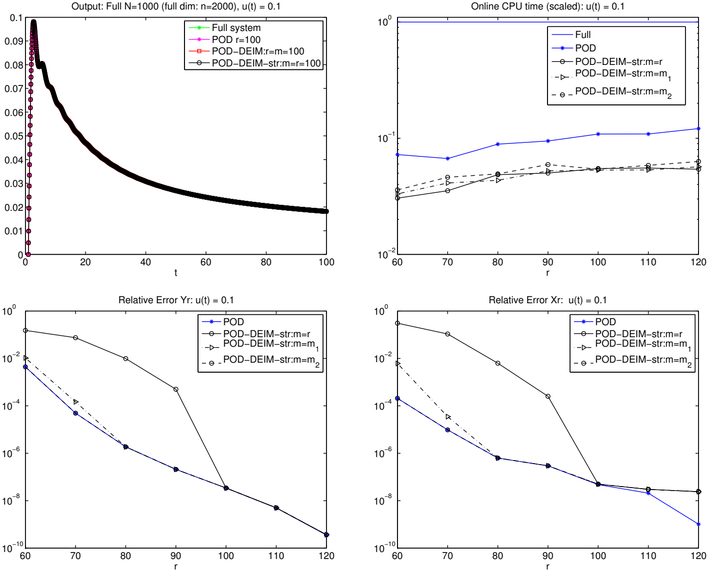






Input: $\mathbf{u}(t) = 0.1$

- POD basis dimension r
- DEIM dim.: $m = r, m_1, m_2$, $m_1 = r + \texttt{ceil}(r/3)$, $m_2 = r + \texttt{ceil}(2r/3)$.



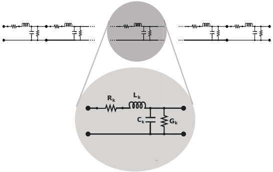
Numerical Test II: N-stage Nonlinear Ladder Network

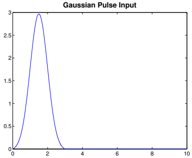
$$\dot{\mathbf{x}} = (\mathbf{J} - \mathbf{R})\nabla_{\mathbf{x}} H(\mathbf{x}) + \mathbf{B}\mathbf{u}(t), \qquad \mathbf{y} = \mathbf{B}^T \nabla_{\mathbf{x}} H(\mathbf{x})$$

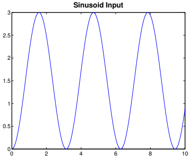
- $\mathbf{J} = \begin{bmatrix} \mathbf{0} & \mathrm{S} \\ -\mathrm{S}^T & \mathbf{0} \end{bmatrix}$ $\mathbf{R} = \begin{bmatrix} G_0\mathbf{I} & 0 \\ 0 & R_0\mathbf{I} \end{bmatrix}$; $\mathbf{B} = [\mathbf{e}_{N+1}, \mathbf{e}_N]$.
- S = upper bidiagonal matrix with 1 on diagonal; -1 on superdiagonal.
- Magnetic fluxes: $\{\phi_k(t)\}_{k=1}^N$; Charges: $\{Q_k\}_{k=1}^N$. $C_k(V) = \frac{C_0 V_0}{V_0 + V}$
- Total energy in stage k: $H^{[k]}(\phi_k, Q_k) = C_0 V_0^2 \left[\exp\left(\frac{Q_k}{C_0 V_0}\right) - 1\right] - Q_k V_0 + \frac{1}{2 L_0}\phi_k^2.$
- State variable: $\mathbf{x} = [Q_1, \ldots, Q_N, \phi_1, \ldots, \phi_N]^T$.
- Hamiltonian:

$$H(\mathbf{x}) = \sum_{k=1}^{N} H^{[k]}(\phi_k, Q_k).$$

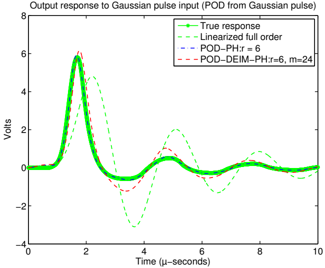
- *Gaussian pulse*-generated POD basis.
- Testing: *Sinusoid input*; $R_0 = 1\Omega$ $G_0 = 10\mu℧$, $L_0 = 2\mu\mathrm{H}$, $C_0 = 100\mathrm{pF}$ $V_0 = 1\mathrm{V}$.

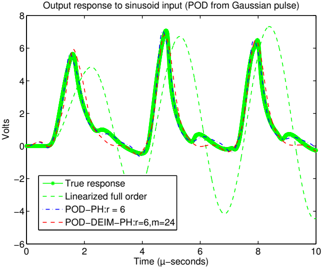


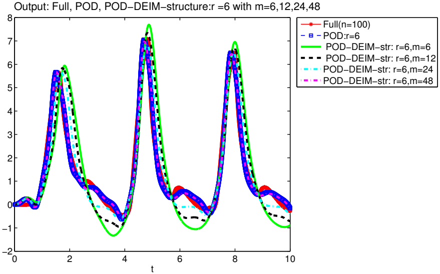




- *Gaussian pulse*-generated POD basis.
- Testing: *Sinusoid input*; $R_0 = 1\Omega$ $G_0 = 10\mu\mho$, $L_0 = 2\mu\text{H}$, $C_0 = 100\text{pF}$ $V_0 = 1\text{V}$.



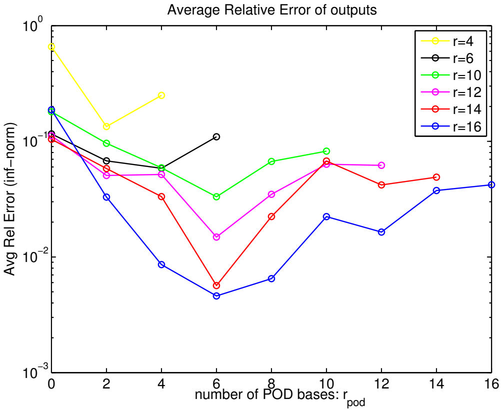




Preliminary results: combining POD and *Quasi-optimal* $\mathcal{H}_2$ bases.

- POD is very accurate for the choice of specific inputs
- Enrich this POD basis by including components that are optimal for (small) variations from an equilibrium point, i.e. optimal subspaces from linear approximations

• Effect of combining *different ratio* of POD and quasi-optimal $\mathcal{H}_2$ basis vectors:



$\Longrightarrow$ Much more accurate than only POD or only quasi-optimal $\mathcal{H}_2$ bases

Compare reduced systems constructed from 3 types of bases: (with/without DEIM)

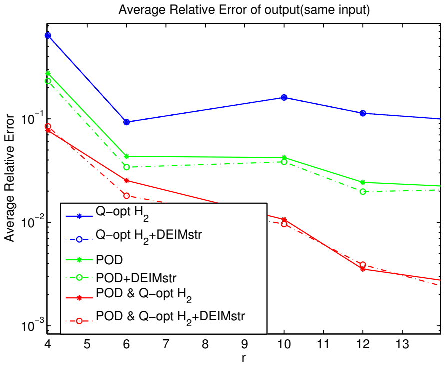
(i) Quasi-optimal $\mathcal{H}_2$ (in blue)

(ii) POD (in green)

(iii) Combined POD & Quasi-optimal $\mathcal{H}_2$ (in red) with ratio $\frac{\#\text{ POD vectors}}{\#\text{ Q-}\mathcal{H}_2\text{ vectors}} = 1$

- Full $n = 100$; Reduced r = $4, 6, \ldots, 14$ (DEIM: $m = 50$).
- The combined basis (iii) is the most accurate.

$r = 12$	Q-$\mathcal{H}_2$	POD	POD & Q-$\mathcal{H}_2$
Rel. Output Error	$\mathcal{O}(10^{-1})$	$\mathcal{O}(10^{-2})$	$\mathcal{O}(10^{-3})$



Conclusions

- Proposed model reduction techniques for nonlinear port-Hamiltonian systems:

  $\Longrightarrow$ Preserve port-Hamiltonian *structure*,
  $\Longrightarrow$ Guaranteed *passivity* and *stability* of reduced systems,
  $\Longrightarrow$ Reduce *complexity* while maintain *accuracy*.

- Based on structure-preserving projection framework (with POD basis)& DEIM
  - Use also *quasi-optimal* $\mathcal{H}_2$ basis (from linear port-Hamiltonian systems [Gugercin *et al.* (2011)]).

- Proposed a-priori error bounds in $\mathcal{L}^2$-norm

  $\Longrightarrow$ Convergence rate is proportional to least-squares errors.

- Efficiency shown through numerical tests (Ladder Network, Toda Lattice).

On-going Work

- A-posteriori error estimate (e.g. [Wirtz, Sorensen, and Haasdonk (2012)])
- Extend to parametrized systems.
- Improve the reduced-order projection subspaces by developing optimal POD and quasi-optimal $\mathcal{H}_2$ combinations.
- Use optimal bilinear spaces from B-IRKA [Benner, Breiten (2012)]

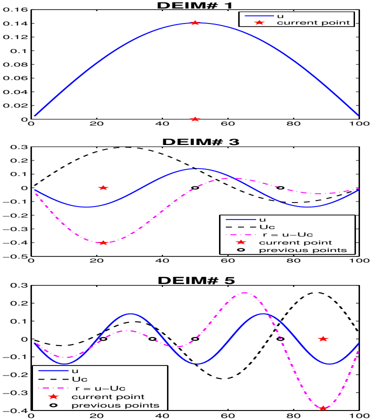
Discrete Empirical Interpolation Method (DEIM) [5]

DEIM

INPUT: $\mathbf{u}_1, \ldots, \mathbf{u}_m \in \mathbb{C}^n$ (linearly independent)

OUTPUT: $\wp_1, \ldots, \wp_m$

- $[\rho \quad \wp_1] = max|\mathbf{u}_1|$
  $\mathbf{U} = [\mathbf{u}_1]$, $\vec{\wp} = [\wp_1]$, $\mathbf{P} = [\mathbf{e}_{\wp_1}]$
- for $j = 2$ to m
  1. $\mathbf{u} \leftarrow \mathbf{u}_j$
  2. Solve $(\mathbf{P}^T\mathbf{U})\mathbf{c} = \mathbf{P}^T\mathbf{u}$ for $\mathbf{c}$
  3. $\mathbf{r} = \mathbf{u} - \mathbf{Uc}$
  4. $[\rho \quad \wp_i] = max\{|\mathbf{r}|\}$
  5. $\mathbf{U} \leftarrow [\mathbf{U} \quad \mathbf{u}]$, $\vec{\wp} \leftarrow \begin{bmatrix} \vec{\wp} \\ \wp_j \end{bmatrix}$, $\mathbf{P} \leftarrow [\mathbf{P} \quad \mathbf{e}_{\wp_j}]$



✠ Discrete variation of the EIM algorithm (Barrault, Maday, Nguyen, Patera; 2004)

[5] S. Chaturantabut and D. C. Sorensen. *Discrete empirical interpolation for nonlinear model reduction*. SIAM J. Sci. Comput., 32(5): pp. 2737-2764, 2010.