Implicit-IMOR method for linear Differential Algebraic Equations

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ModRed2013

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MII
Mathematics for Industrial Innovation

TU/e Technische Universiteit Eindhoven University of Technology

Where innovation starts

Work of PhD student Nicodemus Banagaaya

Jan-Nov: MOR!! (all day, all night)

Nov-Jan!

Kampala: sunny, 26 centigrade

Eindhoven, Magdeburg: dull, foggy, 0-5 centigrade

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Introduction -

Review of IMOR method

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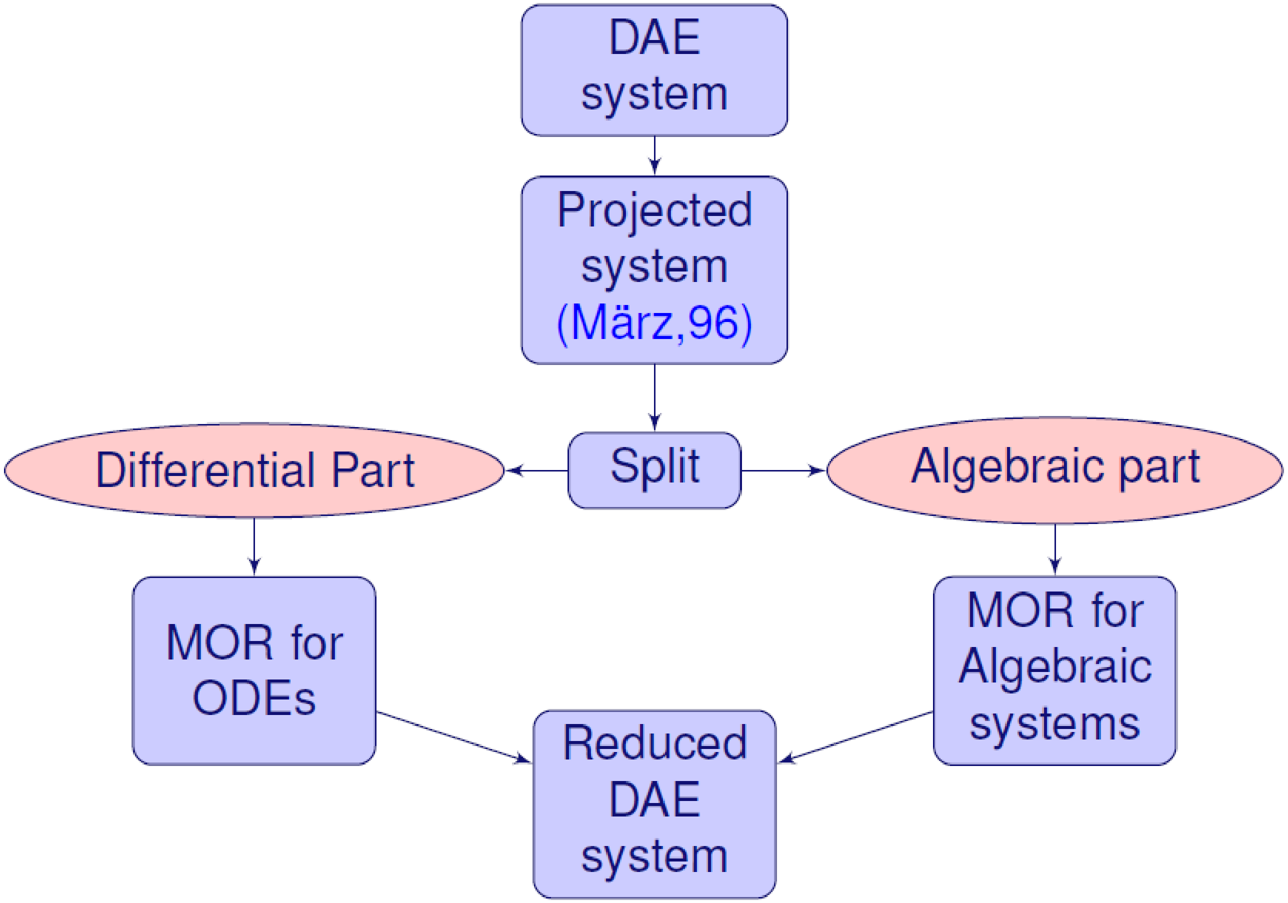
General idea of Index-aware Model Order Reduction (IMOR)



Differential algebraic systems

$$\mathbf{E}x' = \mathbf{A}x + \mathbf{B}u, \quad x(0) = x_0,$$

with **E** singular.

Assumptions:

- Solvability: $\det(\lambda\mathbf{E} - \mathbf{A}) \neq 0$ for some $\lambda \in \mathbb{C}$.
- Input vector: u must be smooth enough.
- Initial conditions: $x(0) = x_0$ must be consistent.

Why did we develop "IMOR"?

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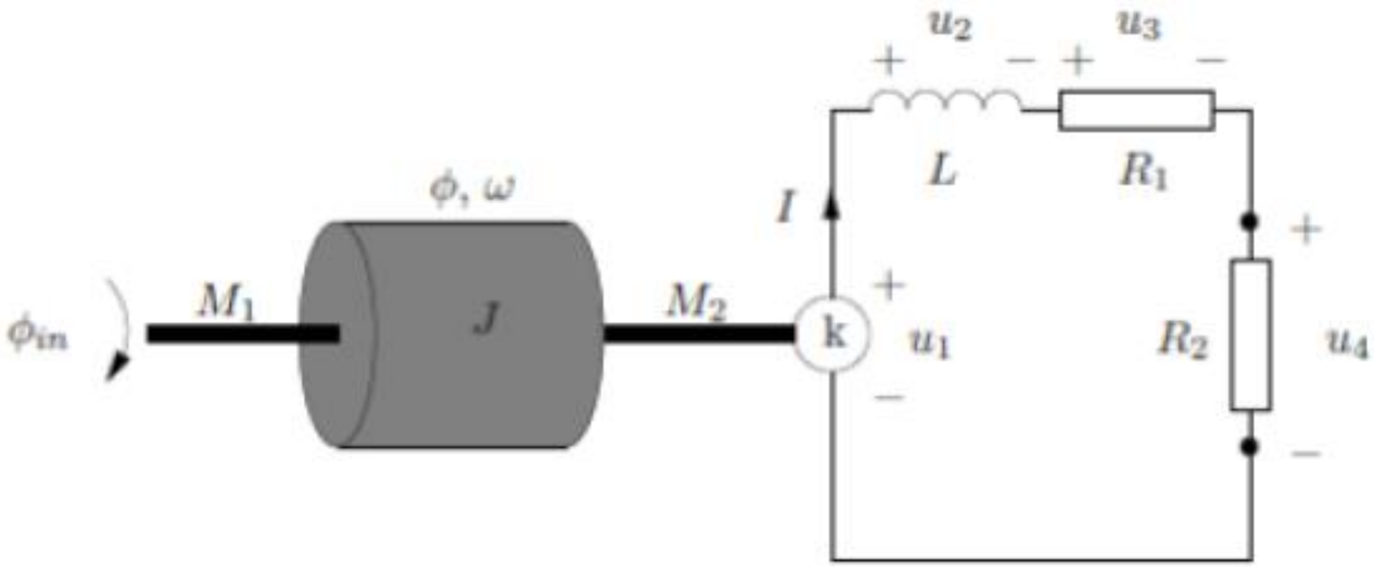
Model of a generator



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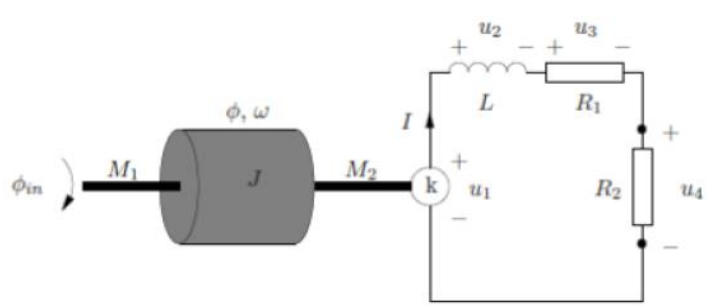
Model of a generator

Find $\mathrm{x} = \begin{bmatrix} M_1 & M_2 & \omega & \phi & \mathrm{I} & u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T$:

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & J & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & L & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & R_2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$



PRIMA reduced order generator model

System is solvable: $\det(\lambda \mathbf{E} - \mathbf{A}) = -R_1 - R_2 - \lambda L \neq 0 \, \forall \lambda \in \mathbb{C}$. Setting $J = 1$, $k = -1$, $R_1 = 1$, $R_2 = 1$, $L = 1$.

PRIMA reduced-order model

$$\mathbf{E}_r = \mathbf{V}_r^T \mathbf{E} \mathbf{V}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2774 & 0.4615 & -0.1155 & 0.0665 \\ -0.0595 & -0.2227 & 0.0557 & -0.0321 \\ -0.2637 & -0.1197 & 0.0299 & -0.0172 \end{bmatrix},$$

$$\mathbf{A}_r = \mathbf{V}_r^T \mathbf{A} \mathbf{V}_r = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -0.2774 & 0.1538 & -0.7175 & 0 \\ -0.8326 & 0.0330 & 0.2944 & -0.0278 \\ 0.4795 & 0.1463 & 0.0962 & -0.0483 \end{bmatrix},$$

$$\mathbf{B}_r = \mathbf{V}_r^T \mathbf{B} = \begin{bmatrix} 0 & 0.2774 & 0.8326 & -0.4795 \end{bmatrix}^T,$$

and $\mathbf{C}_r = \mathbf{V}_r^T \mathbf{C} = \begin{bmatrix} 0 & 0.2774 & 0.0595 & 0.2637 \end{bmatrix}^T$.

$\det(\lambda \mathbf{E}_r - \mathbf{A}_r) = 0 \Rightarrow$ PRIMA model is unsolvable.

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Index-aware MOR needed

- PRIMA may run into problems for higher index systems
- Besides, we feel that it is always good to mimic the structure and properties of the original problem
- **Mimetic methods** are gaining popularity, but have been developed for a long time:
  - Exponentially fitted schemes for singularly perturbed and stiff differential equations
  - Modified ICCG method for iterative solution of linear systems
  - MOR for port-Hamiltonian systems
- As the basis for our IMOR method, we use a method developed in the 1990's

März decoupling procedure

Tractability index

Set $\mathbf{E}_0 := \mathbf{E}$, $\mathbf{A}_0 := \mathbf{A}$, then

$$\mathbf{E}_{j+1} = \mathbf{E}_j - \mathbf{A}_j \mathbf{Q}_j, \quad \mathbf{A}_{j+1} := \mathbf{A}_j \mathbf{P}_j, \quad j \geq 0,$$

where we choose projector $\mathbf{Q}_j$ such that $\mathrm{Im}\mathbf{Q}_j = \mathrm{Ker}\mathbf{E}_j$, $\mathbf{P}_j = \mathrm{I} - \mathbf{Q}_j$. $\exists\, \mu$ such that $\det(\mathbf{E}_\mu) \neq 0$ and all $\det(\mathbf{E}_j) = 0$ for all $0 \leq j < \mu$.
$\mu =$tractability index.

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März decoupling procedure

Projected DAE system

$$\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0 \mathrm{x}' + \mathbf{Q}_0 \mathrm{x} + \cdots + \mathbf{Q}_{\mu-1} \mathrm{x} = \mathbf{E}_{\mu}^{-1} \left(\mathbf{A}_{\mu} \mathrm{x} + \mathbf{B} \mathrm{u} \right),$$

with constraint: $\mathbf{Q}_j \mathbf{Q}_i = 0,\ j > i$,for $\mu > 1$.

Modification of decoupling procedure

Basic idea: Rank-Nullity theorem.

Index 1 systems

Let $n_p = \text{rank}(\mathbf{E}_0)$, $k_0 = n - n_p$. $(\mathbf{p}, \mathbf{q}) = (\mathbf{p}_1, \cdots, \mathbf{p}_{n_p}, \mathbf{q}_1, \cdots, \mathbf{q}_{k_0}) \in \mathbb{R}^n$.
$(\mathbf{p}, \mathbf{q})^{-1} = (\mathbf{p}^*, \mathbf{q}^*)^T$, where $\mathbf{q}^{*T} \in \mathbb{R}^{k_0, n}$ and $\mathbf{p}^{*T} \in \mathbb{R}^{n_p, n}$.

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Modified index-1 system

$$\mathbf{E}x' = \mathbf{A}x + \mathbf{B}u, \; x(0) = x_0$$
$$y = \mathbf{C}^T x.$$

$$\xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p u, \; \xi_p(0) = \mathbf{p}^{*T} x(0),$$

$$\xi_{q,0} = \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} u,$$
$$y = \mathbf{C}_p^T \xi_p + \mathbf{C}_{q,0}^T \xi_{q,0}.$$

Output-Transfer function

$$\mathbf{Y}(s) = \left[\mathbf{H}_p(s) + \mathbf{H}_{q,0}(s)\right] \mathbf{U}(s) \Rightarrow \mathbf{Y}(s) = \mathbf{H}(s)\mathbf{U}(s).$$

IMOR-1 method – descriptor form

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_p} & 0 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{E}}} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix}' = \underbrace{\begin{bmatrix} \mathbf{A}_p & 0 \\ \mathbf{A}_{q,0} & -\mathbf{I}_{k_0} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_{q,0} \end{bmatrix}}_{\tilde{\mathbf{B}}} \mathbf{u},$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{C}_p^T & \mathbf{C}_{q,0}^T \end{bmatrix}}_{\tilde{\mathbf{C}}^T} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix}.$$

Approximate solutions of IMOR:

$$\begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix} = \overbrace{\begin{bmatrix} \mathbf{V}_{p_r} & 0 \\ 0 & \mathbf{V}_{q_{\tau_0}} \end{bmatrix}}^{\mathbf{V}} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_0}} \end{bmatrix}, \quad \mathbf{V}_{q_{\tau_0}} = \mathrm{orth}(\mathrm{span}\{\mathbf{B}_{q,0}, \mathbf{A}_{q,0}\mathbf{V}_{p_r}\})$$

IMOR-1 method – reduced order form

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_{p_r}} & 0 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{E}}_r} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_0}} \end{bmatrix}' = \underbrace{\begin{bmatrix} \mathbf{A}_{p_r} & 0 \\ \mathbf{A}_{q_{\tau_0}} & -\mathbf{I}_{\tau_0} \end{bmatrix}}_{\tilde{\mathbf{A}}_r} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_0}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}_{p_r} \\ \mathbf{B}_{q_{\tau_0}} \end{bmatrix}}_{\tilde{\mathbf{B}}_r} u,$$

$$\mathbf{y}_r = \underbrace{\begin{bmatrix} \mathbf{C}_{p_r}^T & \mathbf{C}_{q_{\tau_0}}^T \end{bmatrix}}_{\tilde{\mathbf{c}}_r^T} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_0}} \end{bmatrix}.$$

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Modified index-2 system

$$\mathbf{E}x' = \mathbf{A}x + \mathbf{B}u, \ x(0) = x_0$$
$$y = \mathbf{C}^T x.$$

$$\xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p u, \ \xi_p(0) = \mathbf{z}_p^{*T} \mathbf{p}^{*T} x(0),$$

$$\xi_{q,1} = \mathbf{A}_{q,1} \xi_p + \mathbf{B}_{q,1} u,$$
$$\xi_{q,0} = \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} u + \mathbf{A}_{q,01} \xi_{q,1}',$$

$$y = \mathbf{C}_p^T \xi_p + \mathbf{C}_{q,1}^T \xi_{q,1} + \mathbf{C}_{q,0}^T \xi_{q,0}.$$

Modified index-2 system

$$\xi_p' = \mathbf{A}_p \xi_p + \mathbf{B}_p \mathrm{u}, \quad \xi_p(0) = \mathbf{z}_p^{*T} \mathbf{p}^{*T} \mathrm{x}(0),$$

$$\xi_{q,1} = \mathbf{A}_{q,1} \xi_p + \mathbf{B}_{q,1} \mathrm{u},$$

$$\xi_{q,0} = \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} u + \mathbf{A}_{q,01} \xi_{q,1}',$$

$$\mathrm{y} = \mathbf{C}_p^T \xi_p + \mathbf{C}_{q,1}^T \xi_{q,1} + \mathbf{C}_{q,0}^T \xi_{q,0}.$$

Output-Transfer function

$$\mathbf{Y}(s) = \left[\mathbf{H}_p(s) + \mathbf{H}_{q,1}(s) + \mathbf{H}_{q,0}(s)\right] \mathbf{U}(s) - \mathbf{C}_{q,0}^T \mathbf{A}_{q,01} \mathbf{B}_{q,1} \mathrm{u}(0),$$

If $\mathbf{A}_{q,01} \mathbf{B}_{q,1} = 0 \Rightarrow \mathbf{Y}(s) = \mathbf{H}(s) \mathbf{U}(s)$. Conventional MOR methods fail if $\mathbf{B}_{q,1} \neq 0$

IMOR-2 method – descriptor form

$$\underbrace{\begin{bmatrix} \mathbf{I}_{n_p} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\mathbf{A}_{q,01} & 0 \end{bmatrix}}_{\tilde{\mathbf{E}}} \begin{bmatrix} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{bmatrix}' = \underbrace{\begin{bmatrix} \mathbf{A}_p & 0 & 0 \\ \mathbf{A}_{q,1} & -\mathbf{I}_{k_1} & 0 \\ \mathbf{A}_{q,0} & 0 & -\mathbf{I}_{k_0} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_{q,1} \\ \mathbf{B}_{q,0} \end{bmatrix}}_{\tilde{\mathbf{B}}} u,$$

$$\mathrm{y} = \underbrace{\begin{bmatrix} \mathbf{C}_p^T & \mathbf{C}_{q,1}^T & \mathbf{C}_{q,0}^T \end{bmatrix}}_{\tilde{\mathbf{C}}^T} \begin{bmatrix} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{bmatrix}.$$

$V_{q_{\tau_1}}$ and $V_{q_{\tau_0}}$ are orthonormal basis matrix of subspaces:

$$\mathcal{V}_{q,1} = \mathrm{span}\{\mathbf{B}_{q,1}, \mathbf{A}_{q,1} V_{p_r}\},$$
$$\mathcal{V}_{q,0} = \mathrm{span}\{\mathbf{B}_{q,0}, \mathbf{A}_{q,01} V_{q_{\tau_1}}, \mathbf{A}_{q,0} V_{p_r}\}.$$

$$\begin{bmatrix} \xi_p \\ \xi_{q,1} \\ \xi_{q,0} \end{bmatrix} = \overbrace{\begin{bmatrix} \mathbf{V}_{p_r} & 0 & 0 \\ 0 & \mathbf{V}_{q_{\tau_1}} & 0 \\ 0 & 0 & \mathbf{V}_{q_{\tau_0}} \end{bmatrix}} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_1}} \\ \xi_{q_{\tau_0}} \end{bmatrix}$$

IMOR-2 method – reduced order form

$$
\underbrace{\begin{bmatrix} \mathbf{I}_{n_r} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\mathbf{A}_{q_{\tau_{01}}} & 0 \end{bmatrix}}_{\tilde{\mathbf{E}}_r} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_1}} \\ \xi_{q_{\tau_0}} \end{bmatrix}' = \underbrace{\begin{bmatrix} \mathbf{A}_{p_r} & 0 & 0 \\ \mathbf{A}_{q_{\tau_1}} & -\mathbf{I}_{\tau_1} & 0 \\ \mathbf{A}_{q_{\tau_0}} & 0 & -\mathbf{I}_{\tau_0} \end{bmatrix}}_{\tilde{\mathbf{A}}_r} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_1}} \\ \xi_{q_{\tau_0}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}_{p_r} \\ \mathbf{B}_{q_{\tau_1}} \\ \mathbf{B}_{q_{\tau_0}} \end{bmatrix}}_{\tilde{\mathbf{B}}_r} u,
$$

$$
\mathrm{y}_r = \underbrace{\begin{bmatrix} \mathbf{C}_{p_r}^T & \mathbf{C}_{q_{\tau_1}}^T & \mathbf{C}_{q_{\tau_0}}^T \end{bmatrix}}_{\tilde{\mathbf{C}}_r^T} \begin{bmatrix} \xi_{p_r} \\ \xi_{q_{\tau_1}} \\ \xi_{q_{\tau_0}} \end{bmatrix}.
$$

Why a new method IIMOR?

- The IMOR method leads to algebraic systems that are **explicit** in the algebraic variables
- This is due to the way the decoupling method is described/constructed
- Not attractive in practice: if we start with a large resistor network (purely algebraic), IMOR would need the inverse of the system matrix
- Question: can we develop a projection method that leads to implicit algebraic systems?
  - so that we can use the methods we developed for the reduction of purely algebraic systems

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The Implicit IMOR method

Delaying the inversion in the decoupling

Original DAE problem

$$\mathbf{E}x' = \mathbf{A}x + \mathbf{B}u, \; x(0) = x_0$$
$$y = \mathbf{C}^T x.$$

Projected DAE system with no inversion

$$\mathbf{E}_\mu \left[\mathbf{P}_{\mu-1} \cdots \mathbf{P}_0 x' + \mathbf{Q}_0 x + \cdots + \mathbf{Q}_{\mu-1} x\right] = \mathbf{A}_\mu x + \mathbf{B}u,$$

with constraint: $\mathbf{Q}_j \mathbf{Q}_i = 0, \; j > i$,for $\mu > 1$.

Implicit index-1 decoupled system

$$\begin{aligned} \mathbf{E}x' &= \mathbf{A}x + \mathbf{B}u, \\ y &= \mathbf{C}^T x. \end{aligned}$$

$$\downarrow$$

$$\begin{aligned} \mathbf{E}_p \xi_p' &= \mathbf{A}_p \xi_p + \mathbf{B}_p u, \\ \mathbf{E}_{q,0} \xi_{q,0} &= \mathbf{A}_{q,0} \xi_p + \mathbf{B}_{q,0} u, \\ y &= \mathbf{C}_p^T \xi_p + \mathbf{C}_{q,0}^T \xi_{q,0}. \end{aligned}$$

where $\mathbf{E}_p = \hat{\mathbf{p}}^T \mathbf{E}\mathbf{p}$, $\mathbf{A}_p = \hat{\mathbf{p}}^T \mathbf{A}\mathbf{p}$, $\mathbf{B}_p = \hat{\mathbf{p}}^T \mathbf{B}$, $\mathbf{E}_{q,0} = -\hat{\mathbf{q}}^T \mathbf{A}\mathbf{q}$, $\mathbf{A}_{q,0} = \hat{\mathbf{q}}^T \mathbf{A}\mathbf{p}$, $\mathbf{B}_{q,0} = \hat{\mathbf{q}}^T \mathbf{B}$ and $\mathrm{span}(\hat{\mathbf{p}}) = \mathrm{Ker}\, \mathbf{q}^T \mathbf{A}^T$, $\mathrm{span}(\hat{\mathbf{q}}) = \mathrm{Ker}\, \mathbf{E}^T$. We note that $\hat{\mathbf{q}} = \mathbf{q}$ if $\mathbf{E}$ is symmetric, $\mathbf{E}_p$ and $\mathbf{E}_{q,0}$ are non-singular.

Descriptor form

Descriptor form

$$\underbrace{\begin{bmatrix} \mathbf{E}_p & 0 \\ 0 & 0 \end{bmatrix}}_{\tilde{\mathbf{E}}} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix}' = \underbrace{\begin{bmatrix} \mathbf{A}_p & 0 \\ \mathbf{A}_{q,0} & -\mathbf{E}_{q,0} \end{bmatrix}}_{\tilde{\mathbf{A}}} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{B}_p \\ \mathbf{B}_{q,0} \end{bmatrix}}_{\tilde{\mathbf{B}}} \mathbf{u},$$

$$\mathbf{y} = \underbrace{\begin{bmatrix} \mathbf{C}_p^T & \mathbf{C}_{q,0}^T \end{bmatrix}}_{\tilde{\mathbf{C}}^T} \begin{bmatrix} \xi_p \\ \xi_{q,0} \end{bmatrix}.$$

Implicit-IMOR reduced order model

$\tilde{\mathbf{E}}_r = \mathbf{W}^T\tilde{\mathbf{E}}\mathbf{V}$, $\tilde{\mathbf{A}}_r = \mathbf{W}^T\tilde{\mathbf{A}}\mathbf{V}$, $\tilde{\mathbf{B}}_r = \mathbf{W}^T\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}_r = \mathbf{V}^T\tilde{\mathbf{C}}$, where

$$\mathbf{V} = \begin{bmatrix} \mathbf{V}_{p_r} & 0 \\ 0 & \mathbf{V}_{q_{\tau_0}} \end{bmatrix}, \quad \mathbf{W} = \begin{bmatrix} \mathbf{V}_{p_r} & 0 \\ 0 & \mathbf{W}_{q_{\tau_0}} \end{bmatrix}$$

$$\mathbf{W}_{q_{\tau_0}} = \mathrm{orth}(\mathrm{span}\{\mathbf{B}_{q,0}, \mathbf{A}_{q,0}\mathbf{V}_{p_r}\}),$$

$$\mathbf{V}_{q_{\tau_0}} = \mathrm{orth}(\mathrm{span}\{\mathbf{E}_{q,0}^{-1}\mathbf{B}_{q,0}, \mathbf{E}_{q,0}^{-1}\mathbf{A}_{q,0}\mathbf{V}_{p_r}\})$$

Construction of bases for projector

The numerical computation of these projectors and their respective bases is feasible and can be done using the sparse LU decomposition- base routine called LUQ. This routine decomposes a singular sparse matrix $\mathbf{E}_0$, into

$$\mathbf{E}_0^T = \mathbf{L}_0 \begin{bmatrix} \mathbf{U}_0 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{R}_0$$

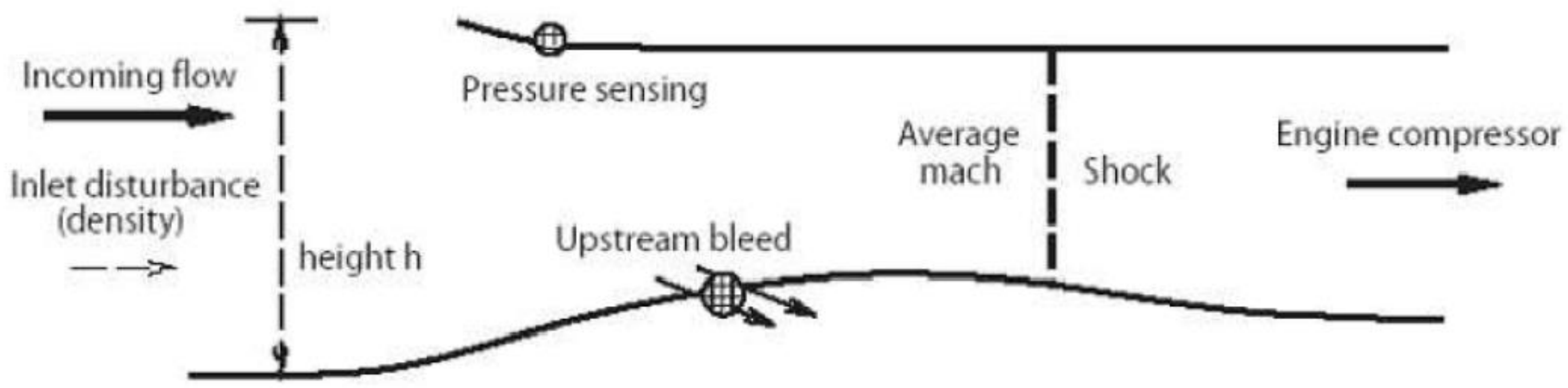
where $\mathbf{L}_0, \mathbf{R}_0 \in \mathbb{R}^{n \times n}$ are nonsingular matrices, $\mathbf{U}_0 \in \mathbb{R}^{r \times r}$ is a nonsingular upper triangular matrix, r is the rank $\mathbf{E}_0$.

[1] P. Kowal (2006, May), Null space of a sparse Matrix. MATLAB Central, http://www.mathworks.co.uk/matlabcentral/fileexchange/11120.

[1] Z. Zhang, N. Wong, An Efficient Projector-Based Passivity Test fot Descriptor Systems, IEEE Trans. On Computer Aided Design of Integrated Circuits And Systems 29(2010), pp. 1203-1214.

Numerical results for IIMOR method

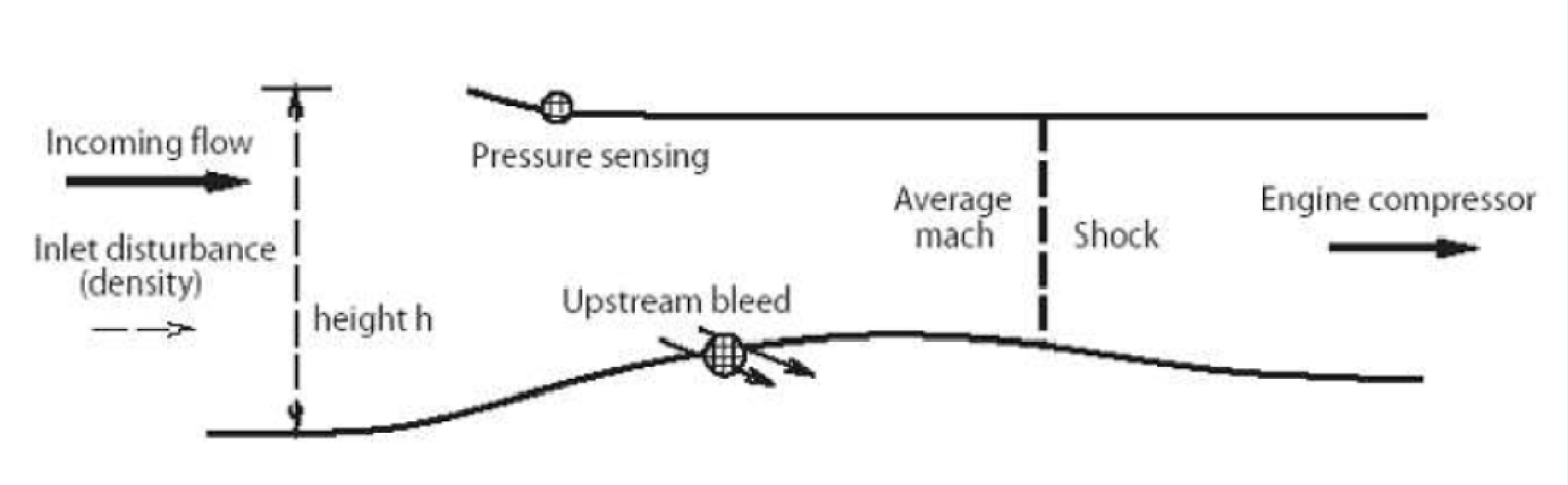
Active Control of a Supersonic Engine Inlet (index-1 problem)



[1] G. Lassaux and K. Willcox, Model reduction of an actively controlled supersonic diffuser, In P. Benner, V. Mehrmann, and D. C. Sorensen, editors, Dimension Reduction of Large- Scale Systems, volume 45 of Lecture Notes in Computational Science and Engineering, pages 357-361. Springer-Verlag, Berlin, Heidelberg, Germany, 2005.

Applications

Active Control of a Supersonic Engine Inlet (index-1 problem)



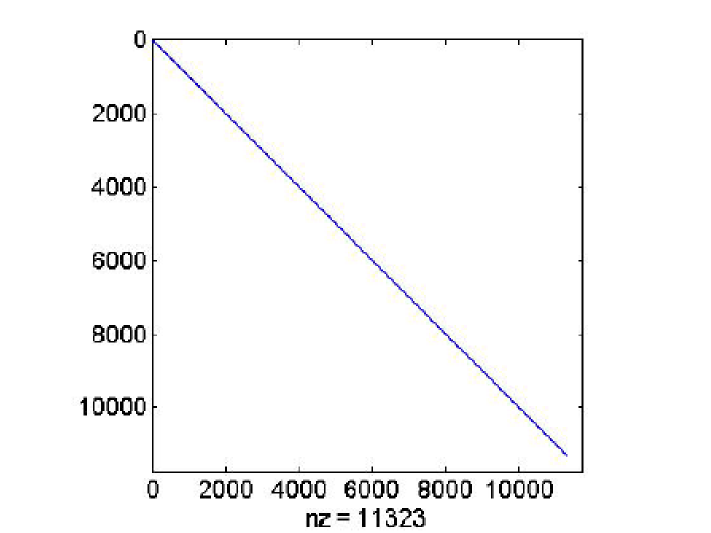
$$\mathbf{E} = \begin{bmatrix} \mathbf{E}_{11} & \mathbf{E}_{12} \\ 0 & 0 \end{bmatrix}, \ \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_1 \\ \mathbf{C}_2 \end{bmatrix}.$$

[1] G. Lassaux and K. Willcox, Model reduction of an actively controlled supersonic diffuser, In P. Benner, V. Mehrmann, and D. C. Sorensen, editors, Dimension Reduction of Large- Scale Systems, volume 45 of Lecture Notes in Computational Science and Engineering, pages 357-361. Springer-Verlag, Berlin, Heidelberg, Germany, 2005.

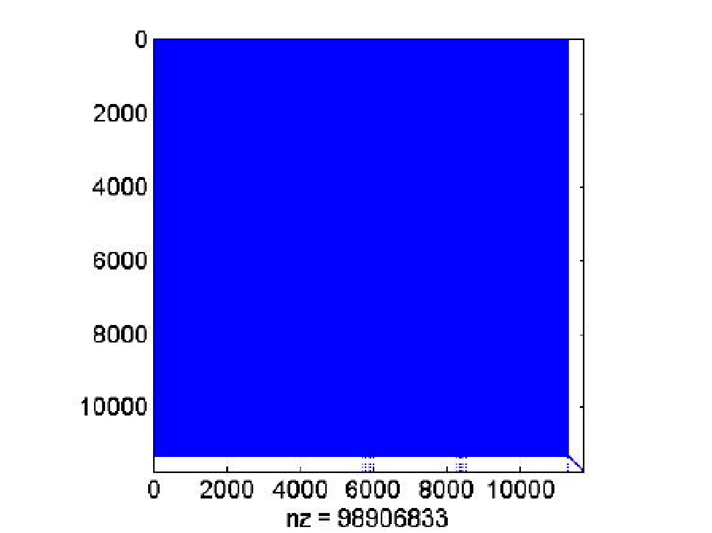
Table : Dimension of decoupled system (n=11730)

Models	Dimension	
	# differential eqns	# Algebraic eqns
Explicit Decoupled Model	11323	407
Implicit Decoupled Model	11323	407

Explict Decoupled Model



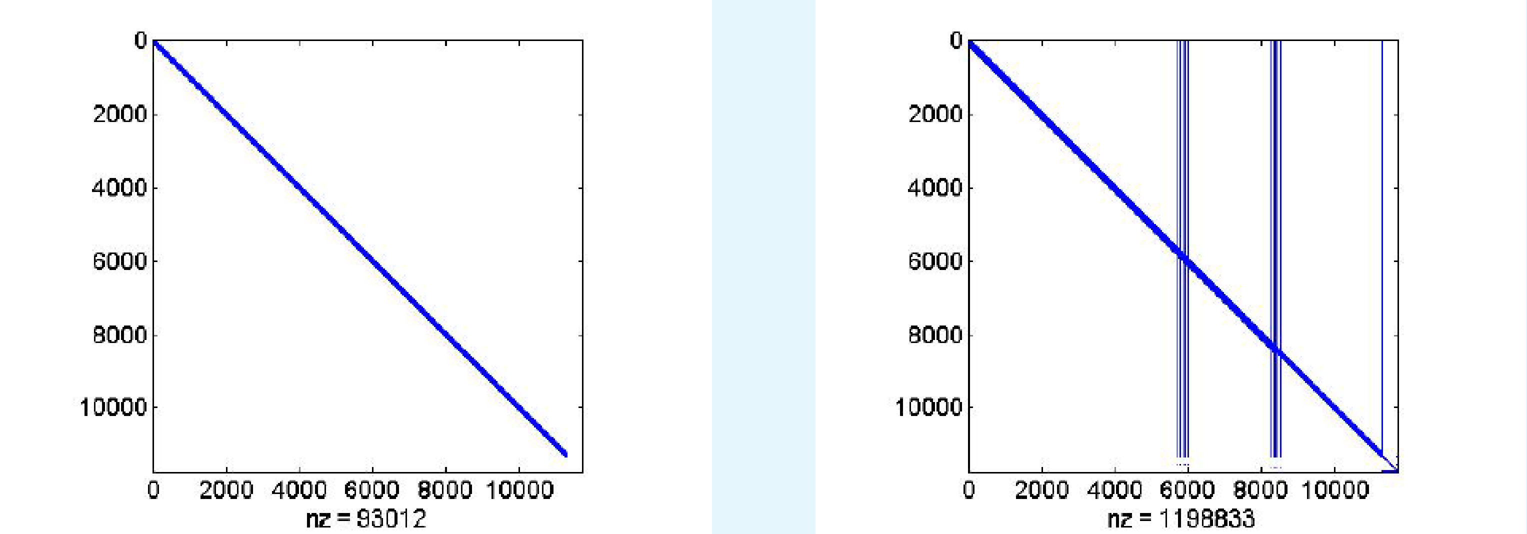
(a) $\tilde{\mathbf{E}}$



(b) $\tilde{\mathbf{A}}$

Figure : Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

Implicit Decoupled Model

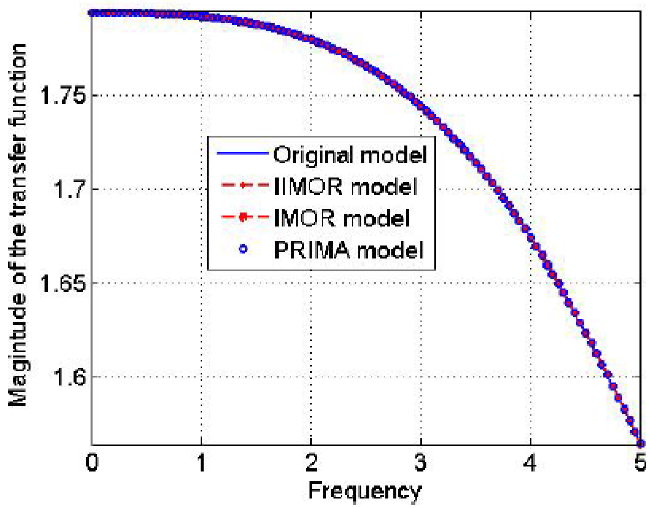


(a) $\tilde{\mathbf{E}}$ (b) $\tilde{\mathbf{A}}$

Figure : Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

Reduced-order model

Models	Dimension	
	# differential eqns	# Algebraic eqns
Original Model	11323	407
IMOR/IIMOR reduced Model	15	16



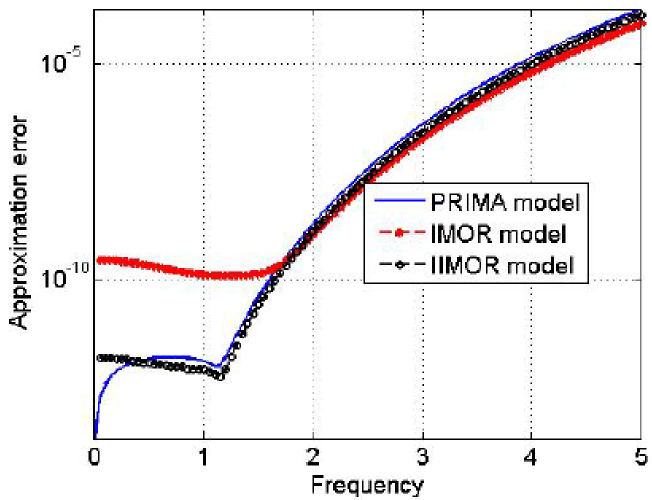
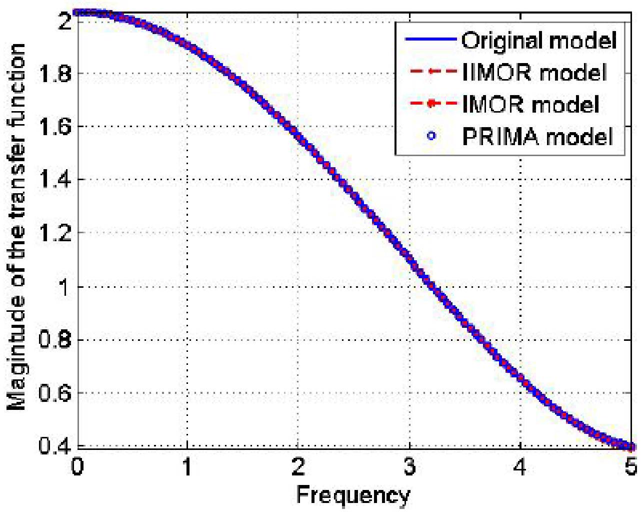


Figure : Transfer function from bleed actuation to average throat Mach number for supersonic diffuser.

Reduced-order model

Models	Dimension	
	# differential eqns	# Algebraic eqns
Original Model	11323	407
IMOR/IIMOR reduced Model	15	16



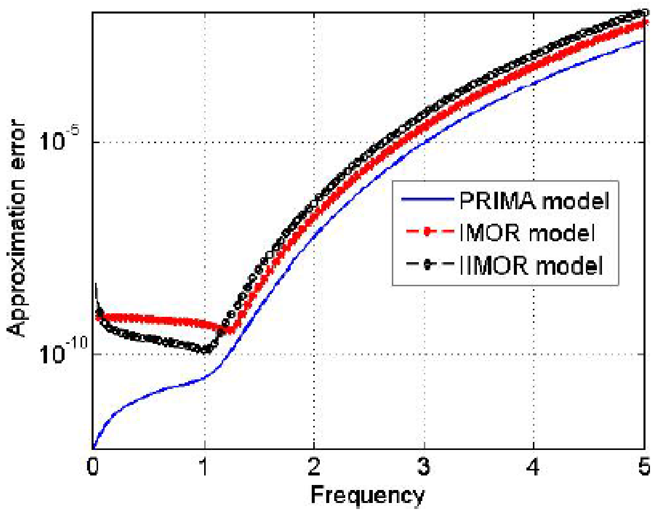


Figure : Transfer function from incoming flow disturbance to average throat Mach number for supersonic diffuser.

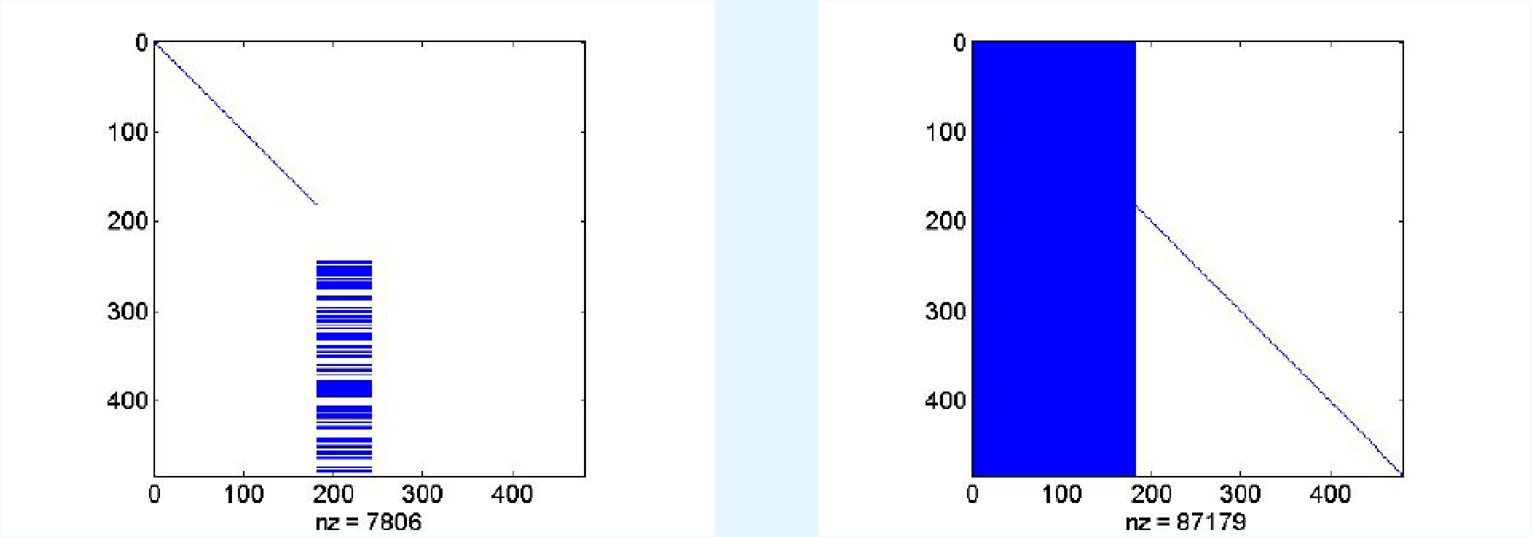
PEEC Model (index-2 probem)

Table : Dimension of decoupled system (n=480)

Models	Dimension		
	# differential eqns	# 1st Algebraic eqns	# 2nd Algebraic eqns
Explicit Decoupled Model	181	61	238
Implicit Decoupled Model	181	61	238

PEEC Model (index-2 probem)

Explict Decoupled Model

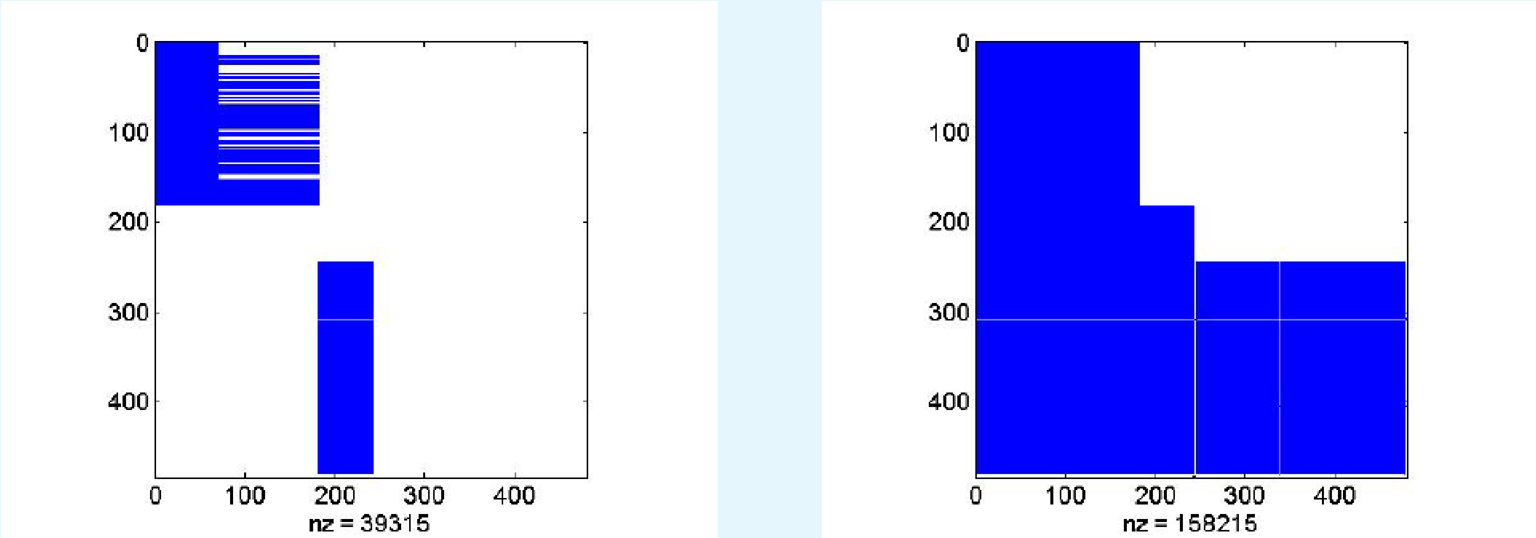


(a) $\tilde{\mathbf{E}}$ (b) $\tilde{\mathbf{A}}$

Figure : Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

PEEC Model (index-2 probem)

Implicit Decoupled Model



(a) $\tilde{\mathbf{E}}$ (b) $\tilde{\mathbf{A}}$

Figure : Sparsity of matrix pencil $(\tilde{\mathbf{E}}, \tilde{\mathbf{A}})$

Reduced-order model

Models	Dimension		
	# differential eqns	# 1st Algebraic eqns	# 2nd Alegbraic eqns
Original Model	181	61	238
IMOR/IIMOR reduced Model	100	2	100

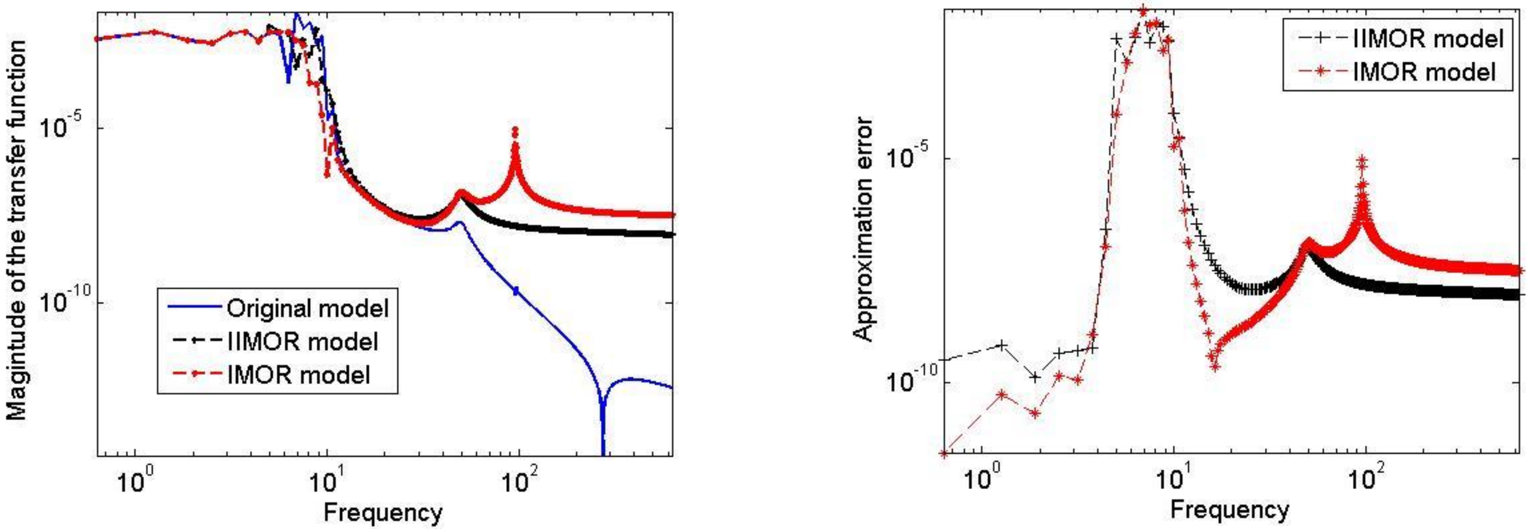
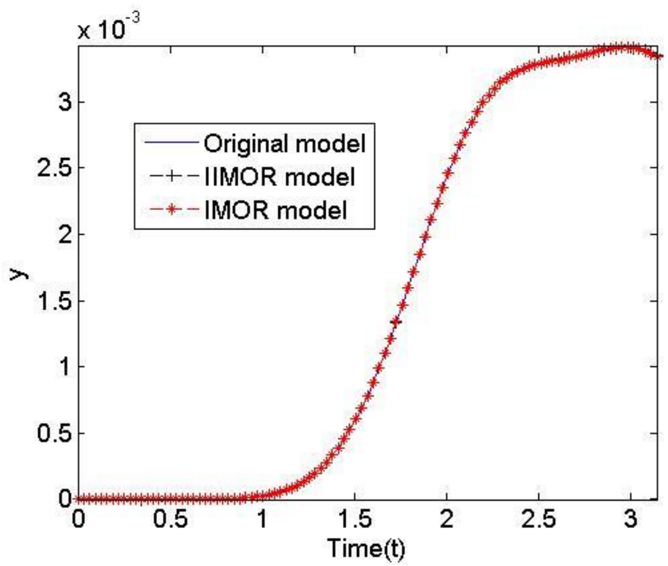


Figure : Comparison of the transfer functions.

Reduced-order model

Models	Dimension		
	# differential eqns	# 1st Algebraic eqns	# 2nd Alegbraic eqns
Original Model	181	61	238
IMOR/IIMOR reduced Model	100	2	100



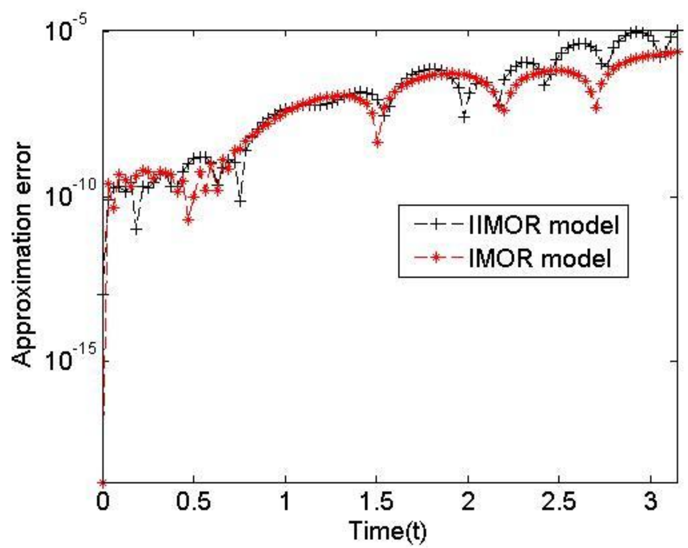


Figure : Comparison of the output solution, $u(t) = \sin(t)$.

Other splitting MOR methods for DAEs

Other MOR methods which also first split the DAE into differential and algebraic parts:

- Serkan Gugercin, Tatjana Stykel, and Sarah Wyatt, Model Reduction of Descriptor Systems by Interpolatory Projection Methods (2013).
- M. Heinkenschloss,D.C. Sorensen, K. Sun. Balanced truncation model reduction for a class of descriptor systems with application to the Oseen equations, SIAM J. Sci. Comput., 30(2):1038-1063, 2008.
- F. D. Freitas, N. Martins, S. L. Varrichio, J. Rommes and F. C. Véliz, Reduced-Order Tranfer Matrices From Network Descriptor Models of Electric Power Grids, IEEE transactions on power systems, 26 (2011), pp. 1905–1919.

Conclusions

Conclusions

- Both decoupling procedure preserves the mathematical properties of the DAE system.
- Implicit-MOR is computationally cheaper and sparser than IMOR method.
- The decoupling techniques developed can also be used to solve DAEs in a robust manner, different from existing methods.

Needed (future work):

Use the methods we developed for purely algebraic systems also in this IIMOR context

(cf paper by Schilders, Marcotte, Shontz in COMPEL, 2012)

References

- W. H. A. Schilders, H. A. Van der Vorst and J. Rommes
  Model Order Reduction: Theory, Research Aspects and Applications
  Springer-Verlag,Berlin Heidelberg, 2008.
- G. Alì, N. Banagaaya, W. H. A. Schilders and C. Tischendorf
  Implicit-IMOR method for index-1 and index-2 linear constant DAEs, In preparation.
- N. Banagaaya and W. H. A. Schilders
  Simulation of electromagnetic descriptor models using projectors,
  Journal of Mathematics in Industry 2013, 3:1.