Technische Universität Braunschweig

A survey about moment matching model order reduction in computational electromagnetism

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Acknowledgement

MoreSim_4_Nano

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- Six different institutes in Germany.
- Four industrial partners. (CST, X-FAB, . . .)

Maxwell's equations

$$\begin{pmatrix} M_\epsilon & 0 \\ 0 & M_\mu \end{pmatrix} \dot{x}(t) = \begin{pmatrix} -M_\sigma & C_H \\ -C_E & 0 \end{pmatrix} x(t) + \mathcal{B}u(t), \; y(t) = \mathcal{C}x(t)$$

- Block structure with respect to electric and magnetic field strength.
- Discretization leads to high-dimensional model problem.
- For model order reduction.
  - Structure-preserving reduced order model.
  - Efficient solution technique for shifted linear systems.

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Outline

- **Moment matching methods in model order reduction**
- **Efficient offline-stage of moment matching methods**
  - Modified adaptive-order rational Arnoldi method
  - Recycling Krylov subspace methods
- **Numerical experiments**
- **Conclusion**

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Moment matching in a nutshell

Taylor expansion

Let $\mathcal{H}(s) = \sum_{j=0}^{\infty} \mathcal{C}X^{(j)}(s_i)(s - s_i)^j$ denote the transfer function at $s_i \in \mathbb{C}$, where $X^{(j)}(s_i) = \left[-(s_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}\right]^{j-1}(s_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{B}$.

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where $X^{(j)}(s_i) =$

Galerkin projection $\Pi = VV^{\mathrm{T}}$ such that

$$\operatorname{span}(V) = \sum_{i=1}^{l} \mathcal{K}_{j_i}(-(s_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}, (s_i\mathcal{E} - \mathcal{A})^{-1}\mathcal{B})$$

where $n_r = j_1 + \ldots + j_l$, $j_i > 0$.

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where $n_r = j_1 + \ldots + j_l$, $j_i > 0$.

Padé approximation

Galerkin projection leads to $\mathcal{H}^{(j)}(s_i) = \tilde{\mathcal{H}}^{(j)}(s_i)$ for all $j = 0, \ldots, j_i - 1$.

Adaptive-order rational Arnoldi method

Let $Y^{(j)}(s_i) \equiv \mathcal{C}X^{(j)}(s_i)$ denote the output moments.

Residual vectors

$$\begin{array}{|c|} \hline r_1(s_1) \\ \hline r_1(s_2) \\ \hline \vdots \\ \hline r_1(s_l) \\ \hline \end{array}$$

V

$$\boxed{\operatorname{argmax}_{s \in \mathcal{S}_l} |Y^{(j)}(s) - \tilde{Y}^{(j)}(s)|}$$

- Note that $r_j(s) = -(s\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}r_{j-1}(s)$ for all $s \in \mathcal{S}_l$.

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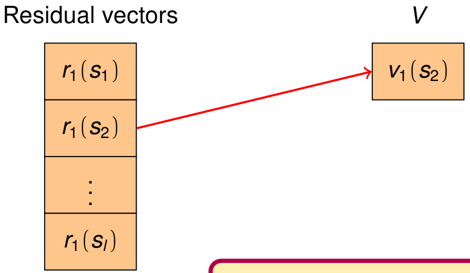

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Residual vectors
$r_1(s_1)$
$\boxed{r_2(s_2)}$
$\vdots$
$r_1(s_l)$

V
$v_1(s_2)$

Modified Gram-Schmidt

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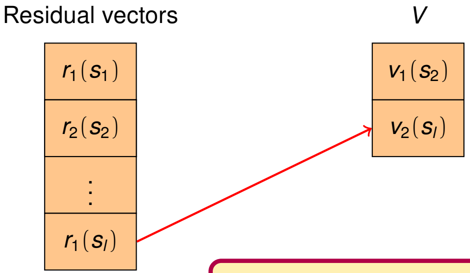

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Adaptive-order rational Arnoldi method

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Residual vectors

$r_1(s_1)$

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$\vdots$

$\boxed{r_2(s_l)}$

V

$v_1(s_2)$

$v_2(s_l)$

Modified Gram-Schmidt

$$\boxed{\operatorname{argmax}_{s \in \mathcal{S}_l} |Y^{(j)}(s) - \tilde{Y}^{(j)}(s)|}$$

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$r_1(s_1)$	$v_1(s_2)$
$r_2(s_2)$	$v_2(s_l)$
$\vdots$	$\vdots$
$r_2(s_l)$	

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Greedy-type expansion point selection: AORA-RK

Lemma ([B., B., 2013])

Let $\mathcal{P}(s) = s\mathcal{E} - \mathcal{A}$ and $\tilde{\mathcal{P}}(s) = s\tilde{\mathcal{E}} - \tilde{\mathcal{A}}$.

$$|Y^{(0)}(s) - \tilde{Y}^{(0)}(s)| \leqslant |\mathcal{C}\mathcal{P}(s)^{-1}| \cdot |h_n(s)| \approx |\tilde{\mathcal{C}}\tilde{\mathcal{P}}(s)^{-1}| \cdot |h_n(s)|.$$

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$$h_n(s_{m+1}) \quad = \quad \mathcal{B} - \mathcal{P}(s_{m+1}) V \tilde{\mathcal{P}}(s_{m+1})^{-1} \tilde{\mathcal{B}}$$

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$$h_n(s_{m+1})\mathbf{u} \quad = \quad \mathcal{B}\mathbf{u} - \mathcal{P}(s_{m+1})V\tilde{\mathcal{P}}(s_{m+1})^{-1}\tilde{\mathcal{B}}\mathbf{u}$$

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$$\begin{aligned} h_n(s_{m+1})\mathbf{u} &= \mathcal{B}\mathbf{u} - \mathcal{P}(s_{m+1})V\tilde{\mathcal{P}}(s_{m+1})^{-1}\tilde{\mathcal{B}}\mathbf{u} \\ \mathcal{P}(s_{m+1})V\tilde{\mathcal{P}}(s_{m+1})^{-1}\tilde{\mathcal{B}}\mathbf{u} &\approx \mathcal{B}\mathbf{u} \end{aligned}$$

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AORA-RK method

Determine $s_{i+1} \in \mathbb{C}$ such that

$$s_{i+1} = \arg\max_{s \in \mathcal{S}} |\tilde{\mathcal{C}}\tilde{\mathcal{P}}(s)^{-1}| \cdot |h_n(s)| \text{ with } \mathcal{S} \subset \imath[f_{\min}, f_{\max}].$$

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Modified generic rational Arnoldi method

Simplified example for two expansion points $s_1 \in \mathbb{C}$ and $s_2 \in \mathbb{C}$.

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$\mathcal{S}_1 = \{s_1\}$

$\boxed{v_1(s_1)}$ $\longrightarrow$ Shifted linear system: $(s_1\mathcal{E} - \mathcal{A})r_j(s_1) = r_{j-1}(s_1)$.

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$v_1(s_1)$

$v_2(s_1)$

Shifted linear system: $(s_1\mathcal{E} - \mathcal{A})r_j(s_1) = r_{j-1}(s_1)$.

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$$\begin{array}{|c|}\hline v_1(s_1) \\ \hline v_2(s_1) \\ \hline \vdots \\ \hline v_{n_r}(s_1) \\ \hline \end{array}$$

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$v_2(s_1)$
$\vdots$
$v_{n_r}(s_1)$

$\mathcal{S}_2 = \{s_1, s_2\}$

$v_1(s_1) \longrightarrow (s_1\mathcal{E} - \mathcal{A})r_j(s_1) = r_{j-1}(s_1)$

$(s_2\mathcal{E} - \mathcal{A})r_j(s_2) = r_{j-1}(s_2)$

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$$\boxed{(s_1\mathcal{E} - \mathcal{A})r_j(s_1) = r_{j-1}(s_1)}$$

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$\vdots$

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$v_3(s_2)$

$v_4(s_1)$

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Expensive approach due to repeated solution of linear systems.

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$$(s_2\mathcal{E} - \mathcal{A})r_j(s_2) = r_{j-1}(s_2)$$

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$$(s_2\mathcal{E} - \mathcal{A})r_j(s_2) = r_{j-1}(s_2)$$

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$v_2(s_1)$	$v_2(s_2)$
$\vdots$	$v_3(s_2)$
$v_{n_r}(s_1)$	$v_4(s_1)$
	$\vdots$
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$$(s_2\mathcal{E} - \mathcal{A})r_j(s_2) = r_{j-1}(s_2)$$

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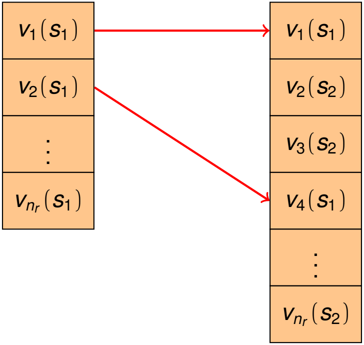

$$(s_2\mathcal{E} - \mathcal{A}) r_j(s_2) = r_{j-1}(s_2)$$

Modified generic rational Arnoldi method

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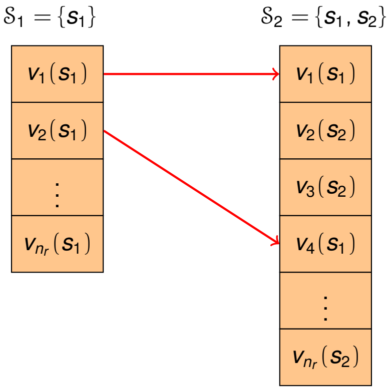

$$(s_2\mathcal{E} - \mathcal{A})r_j(s_2) = r_{j-1}(s_2)$$

- Significantly smaller number of solutions to linear systems.
- Consideration of the order of previous orthonormal vector sequence.
- Extension to multiple expansion points straightforward.

Modified adaptive-order rational Arnoldi method

Compute sequence of reduced order models with $\mathcal{S}_{i+1} = \mathcal{S}_i \cup \{s_{i+1}\}$, $s_{i+1} \in \mathbb{C}$.

$\mathcal{S}_i$ $\mathcal{S}_{i+1}$ $r_j = -(s_{i+1}\mathcal{E} - \mathcal{A})^{-1}\mathcal{E}r_{j-1}$

$v_1(s_1)$
$v_2(s_2)$
$v_3(s_3)$
$\vdots$

$r_1(s_{i+1})$

$$\text{argmax}_s \, |Y^{(j)}(s) - \tilde{Y}^{(j)}(s)|$$

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$\mathcal{S}_{i+1}$

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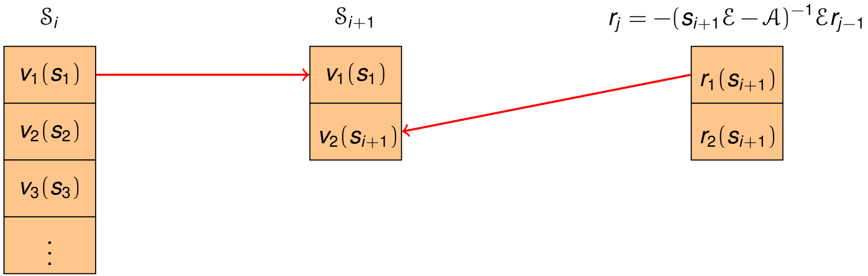

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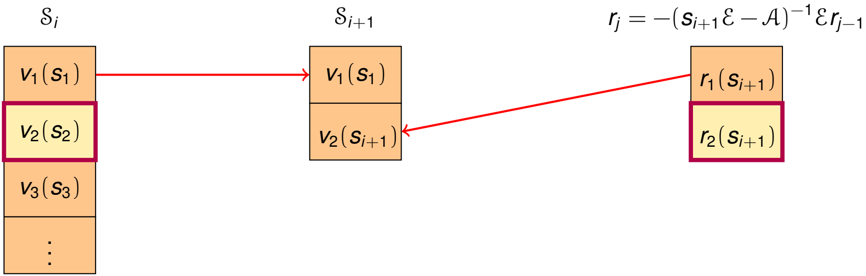

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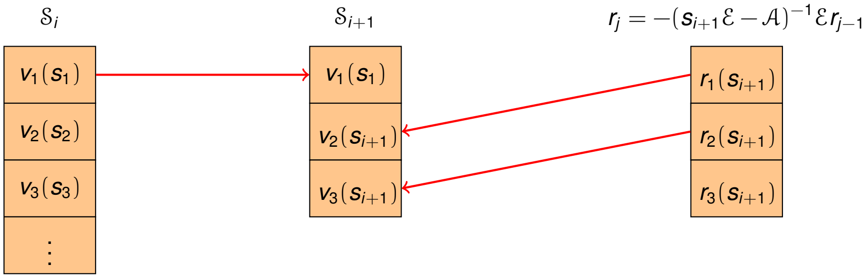

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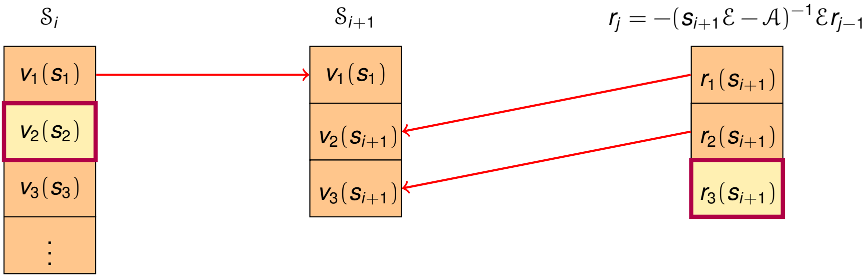

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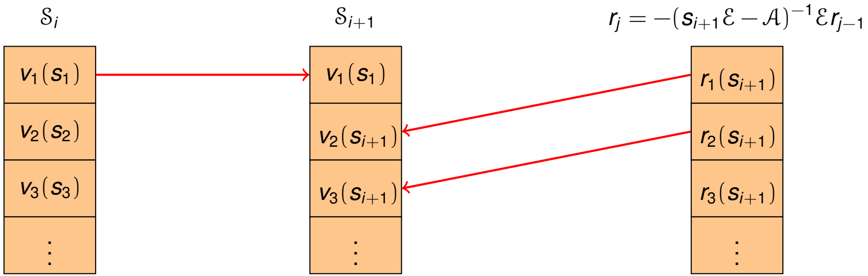

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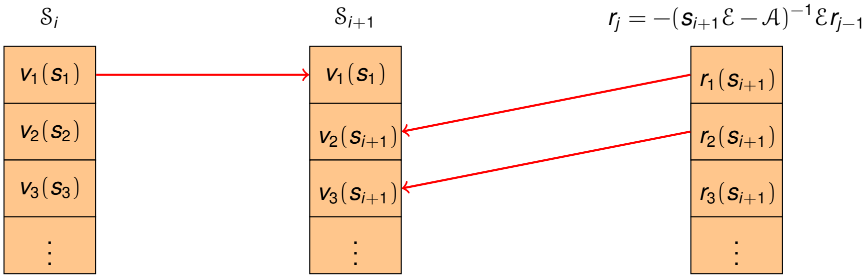

$$\boxed{\operatorname{argmax}_s |Y^{(j)}(s) - \tilde{Y}^{(j)}(s)|}$$

- Additionally consider expansion points $s_l \in \mathcal{S}_i$ with $j_{l,i+1} > j_{l,i}$.

Outline

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- **Efficient offline-stage of moment matching methods**
  - Modified adaptive-order rational Arnoldi method
  - Recycling Krylov subspace methods
- **Numerical experiments**
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Motivation

Schur complement with J-symmetry:

$$S_i = (s_i M_\epsilon + M_\sigma) + C_E (s_i M_\mu)^{-1} C_H \quad \text{with} \quad S_i^{\mathrm{T}} J = J S_i \quad \text{and} \quad J = I.$$

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$$x_k = x_0 + \mathcal{K}_k(S_i, r_0) \quad \text{such that} \quad r_k = f - S_i x_k \perp \mathcal{L}_k$$

Either employ $\mathcal{L}_k \equiv \mathcal{K}_k(S_i, r_0)$ or $\mathcal{L}_k \equiv \mathcal{K}_k(S_i^{\mathrm{T}}, \tilde{r}_0)$.

SQMR method

Determine $x_k = x_0 + V_k y_k$ with minimization

$$\|\rho_0 e_1 - \underline{T}_k y_k\| = \min_y \|\rho_0 e_1 - \underline{T}_k y\|.$$

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Unsym. Lanczos method.

$$\begin{aligned} S_i V_k &= V_{k+1} \underline{T}_k \\ S_i^{\mathrm{T}} W_k &= W_{k+1} \underline{\tilde{T}}_k \end{aligned}$$

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Inefficient for multiple solution to different shifted linear systems and multiple right hand sides.

Recycling SQMR method

Multiple solution to linear system $S_j x_j = f_j$ with $f_j \in \mathbb{C}^{n \times p}$ and $j = 1, \ldots, l$.

Recycling Krylov subspace methods

- Solution update:

$$x_k = x_0 + V_k y_k + U_j z$$

- Arnoldi-/Lanczos-type method:

$$(I - C_j(\tilde{C}_j^{\mathrm{T}} C_j)^{-1} \tilde{C}_j^{\mathrm{T}}) S_j V_k = V_{k+1} \underline{T}_k$$

- Biorthogonality condition:

$$\left[\tilde{C}_j, W_k\right] \perp_b [C_j, V_k]$$

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Employing (Harmonic) Ritz values, it follows that
$$U_j = [U_{j-1}, \bar{V}_k].$$

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- $J = J^{\mathrm{T}}$: $\tilde{U}_j = J U_j$, $\tilde{C}_j = J C_j$
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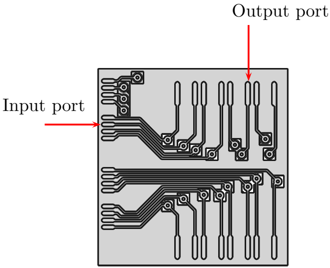
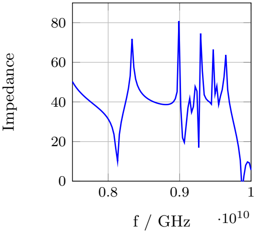
PCB circuit



(a) Model problem.



(b) Transfer function.

Figure: PCB circuit.

- Frequency range: $[f_{\min}, f_{\max}] = [7.5, 10.0]$ GHz.
- Electric conductivity: $\sigma \equiv 0$.
- Dimension of original model problem: $n = 226458$.

AORA-RK vs. AORA-MAX vs. AORA-H2

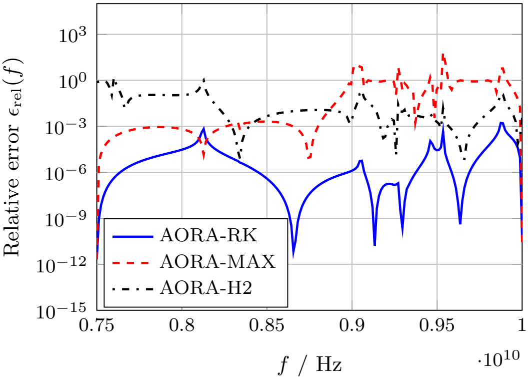
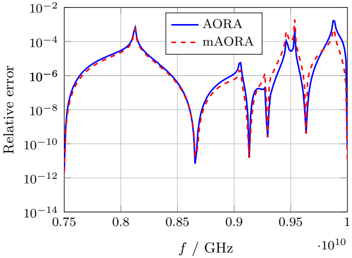


Figure: PCB circuit: $n = 226458 \rightarrow n_r = 20(80)$.

PCB circuit: mAORA method



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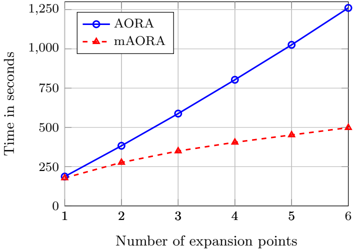

(b)

Figure: PCB circuit: Modified adaptive-order rational Arnoldi method. $(n_r = 20)$

PCB circuit: mAORA method with rSQMR

Expansion point	# rSQMR	# SQMR
$s_1 = 5.44\text{e+}10\imath$	–	–
$s_2 = 6.28\text{e+}10\imath$	54, 24, 25, 24, 25, 24, 24, 24, 25, 25, 25	55
$s_3 = 6.05\text{e+}10\imath$	23, 20, 19, 20, 20, 20, 20, 20, 19	50
$s_4 = 5.84\text{e+}10\imath$	16, 13, 13, 13, 13, 13, 13	35
$s_5 = 4.71\text{e+}10\imath$	23, 21, 21, 18, 21	29
$s_6 = 5.73\text{e+}10\imath$	25, 11, 11, 10, 11	26

Table: PCB circuit: Iteration steps of rSQMR method in mAORA method.

- Single preconditioning technique via
$$S = (s_* M_\epsilon + M_\sigma) + C(s_* M_\mu)^{-1} C^{\mathrm{T}} \quad \text{with} \quad s_* = \imath\sqrt{f_{\min} f_{\max}}.$$
- Significantly smaller number of matrix-vector multiplications.

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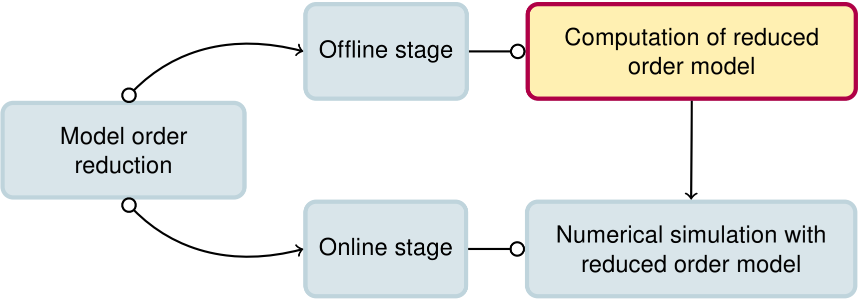
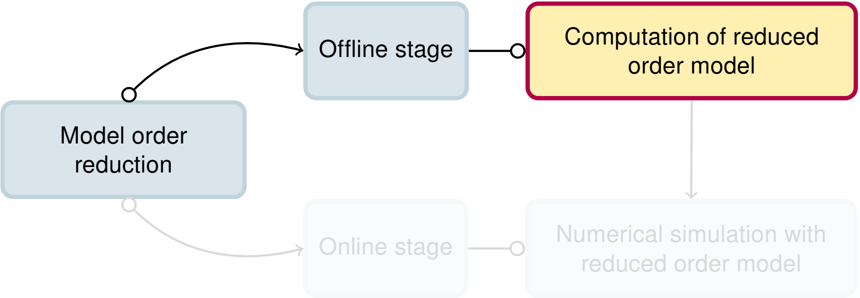
Conclusion



- Structure preservation.
- AORA-RK method.
- Divergence preservation.

- Modified rational Arnoldi method.
- Direct solver: ATLM.
- rSQMR method.

Conclusion



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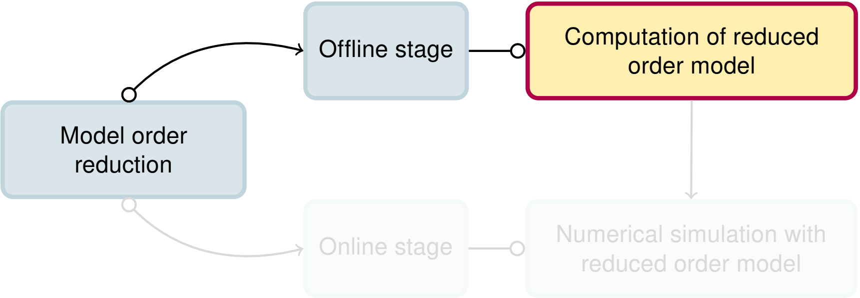

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Offline stage

Computation of reduced order model

Model order reduction

ation with

r model

Thank you for your attention!

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