Model Reduction of Complex Dynamical Systems 2013
December 11, 2013, MPI Magdeburg

Linear Subspace Reduction for Quasistatic Field Simulations

Daniel Schmidthäusler[1], Sebastian Schöps[2], Markus Clemens[1]

[1]Bergische Universität Wuppertal, Chair of Electromagnetic Theory

42119 Wuppertal, Germany

[2]Technische Universität Darmstadt, Graduate School Computational Engineering (CE)

64293 Darmstadt, Germany

Outline

- **Motivation**
- **Quasistatic Fields**
  - –Slowly varying Fields
  - –Quasistatic Field Approximation
  - –System Sub-structuring
- **Linear Subproblem Model Order Reduction**
  - –Proper Orthogonal Decomposition
  - –Magneto-Quasistatic Example
- **Uncertainty Quantification using MOR**
  - –Stochastic Setup
  - –Electro-Quasistatic Example
- **Summary**

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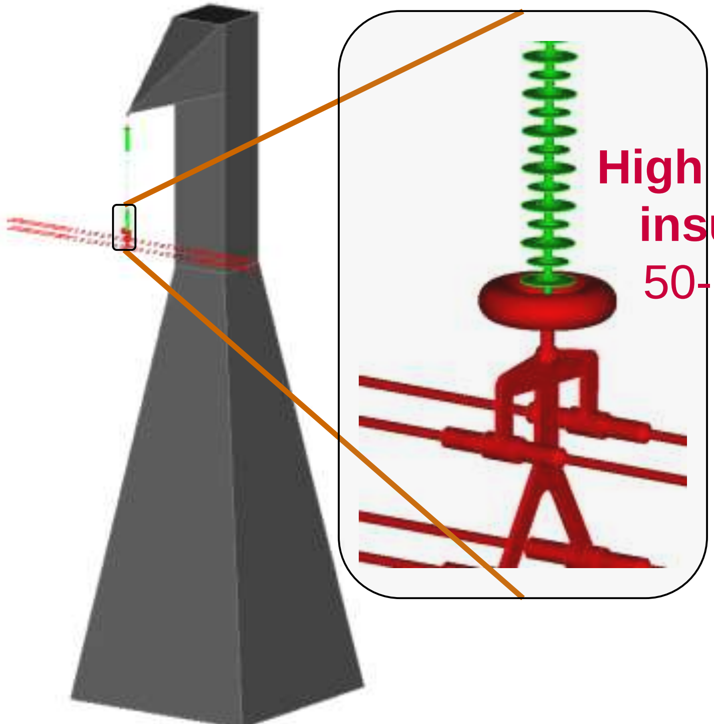
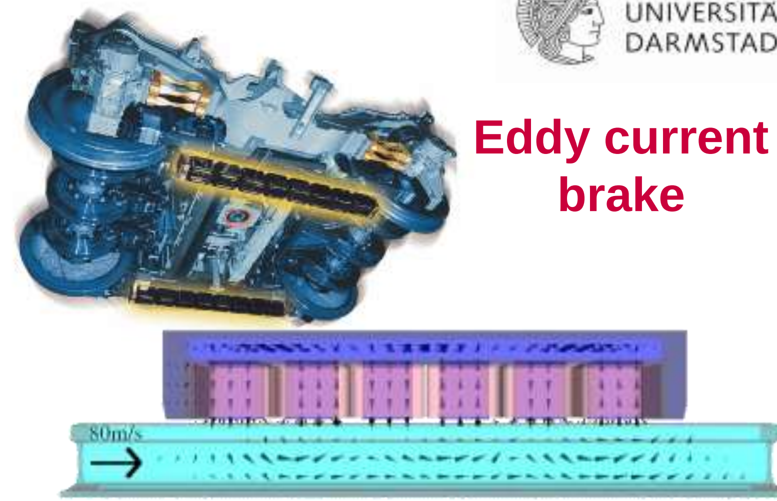
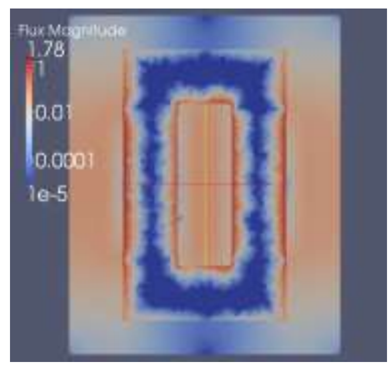
Slowly Varying Electromagnetic Fields

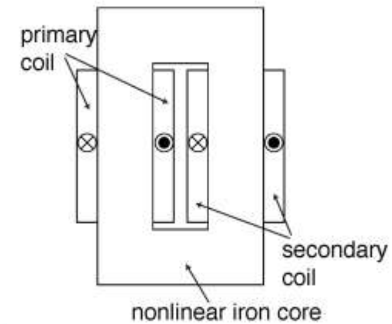


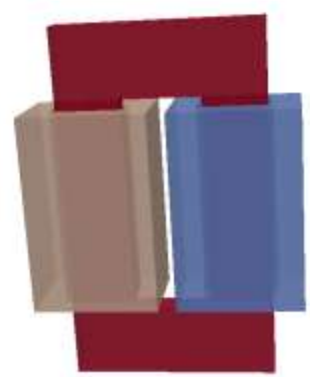
High voltage insulator
50-60 Hz



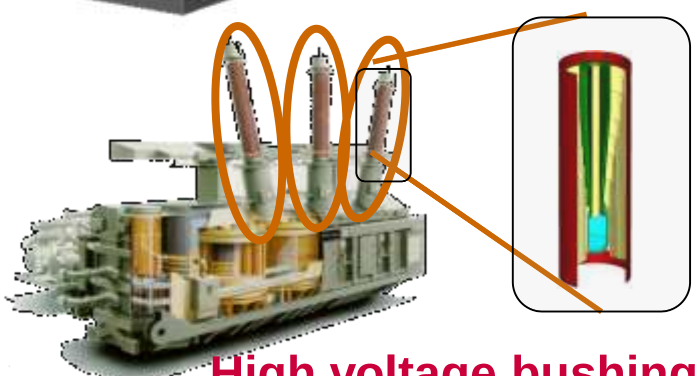
Eddy current brake







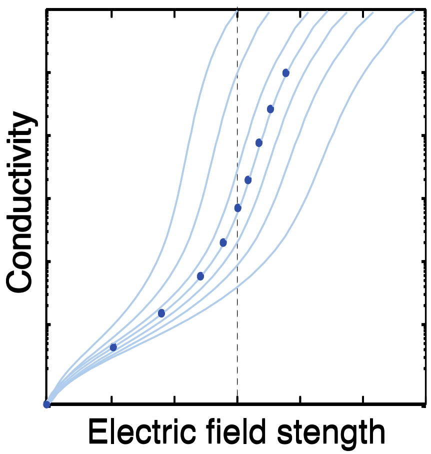
Transformator
50-60 Hz

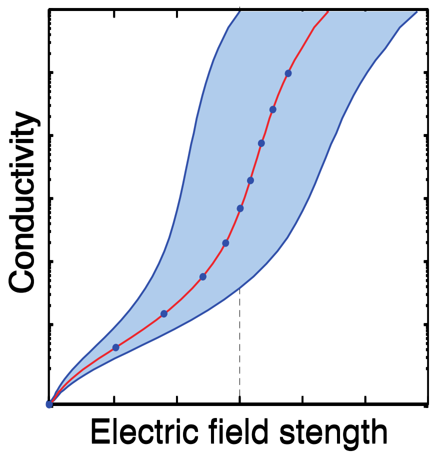


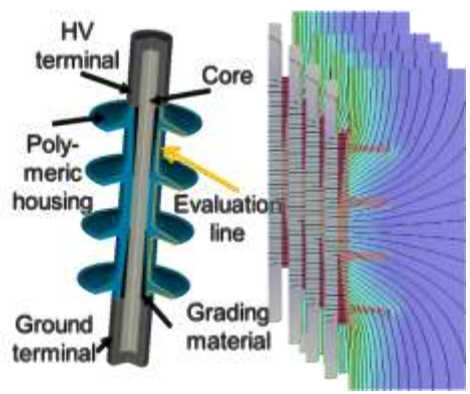
High voltage bushing
50-60 Hz

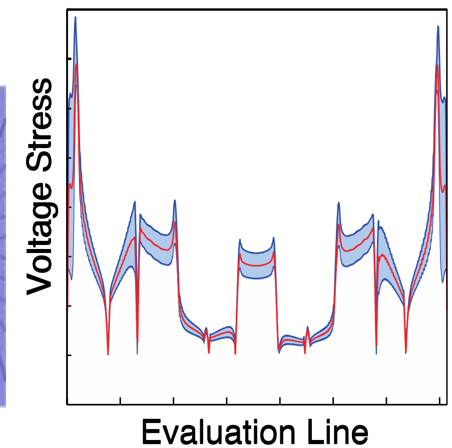
Motivation: Simulation Chain

Uncertain Input → Stochastic Modeling → Multiple Simulations → Uncertain Output









cf. Ulrich Römer et al.: A First Order k-th Moment Method for the Nonlinear Eddy Current Problem with Material Uncertainties

Quasistatic Field Approximation

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Slowly Varying Field Approximations

Neglecting inductive effects:

$$\frac{\partial}{\partial t}\vec{B} \equiv 0$$

Electro-Quasistatic EQS

Neglecting capacitive effects:

$$\frac{\partial}{\partial t}\vec{D} \equiv 0$$

Magneto-Quasistatic MQS

Applied to **Maxwell** equations

Electric scalar potential formulation

$$\frac{\partial}{\partial t}\nabla\cdot(\varepsilon\,\nabla\phi(t))+\nabla\cdot(\kappa(\phi)\nabla\phi(t))=0$$

Magnetic vector potential formulation

$$\kappa\frac{\partial}{\partial t}\vec{A}(t)+\nabla\times(\nu(\vec{A})\nabla\times\vec{A}(t))=\vec{J}_s(t)$$

➔ Initial-boundary problems in time domain

State-of-the-Art

Applying spatial discretizations (FEM, FIT,…)

Electro-Quasistatic

$$\mathbf{M}_{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}+\mathbf{K}_{\kappa}(\boldsymbol{\phi})\boldsymbol{\phi}=\mathbf{b}$$

Magneto-Quasistatic

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}+\mathbf{K}(\mathbf{a})\mathbf{a}=\mathbf{j}_{s}$$

Two Solution Strategies:

- **„Brute Force Approach“** ➔ **Faster Solvers:** ManyCore-/GPU-computing, improved algebraic system solvers, novel formulations,…
- **„Condensate the Problem“** ➔ Equivalent circuit models, **Model Order Reduction**

State-of-the-Art

Applying spatial discretization (FEM, FIT,…)

Electro-Quasistatics

$$\mathbf{M}_{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}+\mathbf{K}_{\kappa}(\boldsymbol{\phi})\boldsymbol{\phi}=\mathbf{b}$$

Magneto-Quasistatics

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}+\mathbf{K}(\mathbf{a})\mathbf{a}=\mathbf{j}_{s}$$

State-of-the-Art in Model Order Reduction for Quasistatics:

- **Albunni et al., 2010:** Trajectory piecewise linearization approach applied for non-linear MQS problem coupled to mechnically moving components.
- **Sato, Igarashi, CEFC 2012:** Eddy current problem reduced by a snapshot approach.
- **Henneron, Clénet, EMF 2013 / COMPUMAG 2013:** Non-linear MQS-circuit coupled problem by modified POD technique (DEIM, Discrete Empirical Interpolation Method)

Time Discretization

After spatial discretization (FEM, FIT,…) ➔

- Time discretization yields a nonlinear system

$$\underbrace{\left(\frac{1}{Dt}\mathbf{M}+\mathbf{K}(f)\right)}_{\mathbf{J}(f)} f = rhs$$

- Application of Newton‘s method

$$\mathbf{F}(f) := \mathbf{J}(f)f - rhs = 0$$

$$\mathbf{J}(f^{(n)})Df^{(n+1)} = -\mathbf{F}(f^{(n)})$$

- Linear systems are solved repeatedly

$$\begin{bmatrix} \mathbf{J}_{11}(f_1) & \mathbf{J}_{12} \\ \mathbf{J}_{21} & \mathbf{J}_{22}\end{bmatrix}\begin{bmatrix} Df_1 \\ Df_2\end{bmatrix} = \begin{bmatrix}\mathbf{F}_1\\ \mathbf{F}_2\end{bmatrix}$$

Typically:
Large constant block

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System Sub-Structuring

Applying spatial discretization (FEM, FIT,…)

Electro-Quasistatics

$$\mathbf{M}_{\varepsilon}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}+\mathbf{K}_{\kappa}(\boldsymbol{\phi})\boldsymbol{\phi}=\mathbf{b}$$

Only **nonlinear** conductive part has relevant conductivity.

Magneto-Quasistatics

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}+\mathbf{K}(\mathbf{a})\mathbf{a}=\mathbf{j}_s$$

Only conductive part has **non-linear** reluctance.

Split systems into **conductive (1)** and **non-conductive (2)** parts.

$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{M}_{12}\\ \mathbf{M}_{21} & \mathbf{M}_{22}\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{bmatrix}+\begin{bmatrix}\mathbf{K}_{11}(\boldsymbol{\phi}) & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\boldsymbol{\phi}_1\\ \boldsymbol{\phi}_2\end{bmatrix}=\begin{bmatrix}\mathbf{b}_1\\ \mathbf{b}_2\end{bmatrix}$$

Stiff nonlinear ODE

→ CEFC 2012, Oita

$$\begin{bmatrix}\mathbf{M}_{11} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix}+\begin{bmatrix}\mathbf{K}_{11}(\mathbf{a}) & \mathbf{K}_{12}\\ \mathbf{K}_{21} & \mathbf{K}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{a}_1\\ \mathbf{a}_2\end{bmatrix}=\begin{bmatrix}\mathbf{j}_1\\ \mathbf{j}_2\end{bmatrix}$$

DAE

→ IGTE 2010, Graz

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Model Order Reduction

Proper Orthogonal Decomposition POD

Assemble system dynamics (solutions „snapshots") $\mathbf{x}^{\alpha} = \mathbf{x}(t_{\alpha})$ in

$\mathbf{X} = [\mathbf{x}^1, \ldots, \mathbf{x}^p]$ „**Snapshot matrix**" of time step solutions

Singular value decomposition (**SVD**)

$$\mathbf{X} = \mathbf{U}\Sigma\mathbf{V}^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_p \mathbf{u}_p \mathbf{v}_p^T \qquad \sigma_i \geq \sigma_{i+1} \geq 0$$

Selecting the reduced basis („How many SVs should be selected?")

$$\mathrm{I}(r) = \frac{\sigma_1 + \cdots + \sigma_r}{\sigma_1 + \cdots + \sigma_r + \sigma_{r+1} + \cdots + \sigma_p} \approx \mathbf{1} \quad \Rightarrow \quad r$$

„**Relative Information Criterion**"

$$\left\| \mathbf{X} - \mathbf{X}_r \right\|_2 = \min_{rank(\mathbf{A}) \leq r} \left\| \mathbf{X} - \mathbf{A} \right\|_2 = \sigma_{r+1}(\mathbf{X})$$

„**2-Norm**"

with $\mathbf{X}_r = \mathbf{U}_r \Sigma \mathbf{V}_r^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$ „**Low-Rank-Approximation**"

Model Order Reduction

Subspace Reduction 1

Representation of essential directions where problem dynamic occurs:

$\mathbf{U}_r = \left[\mathbf{u}^1, \ldots, \mathbf{u}^r\right]$ ➔ $\hat{\mathbf{x}} = \mathbf{U}_r \mathbf{z}$ low order approx. of $\mathbf{x}$

System projection

$$\mathbf{M}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x} + \mathbf{K}(\mathbf{x})\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad \mathbf{U}_r^{\mathrm{T}}\mathbf{M}\mathbf{U}_r\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{z} + \mathbf{U}_r^{\mathrm{T}}\mathbf{K}(\mathbf{z})\mathbf{U}_r\mathbf{z} = \mathbf{U}_r^{\mathrm{T}}\mathbf{b}$$

with the projector $\mathbf{P}$ given by $\hat{\mathbf{x}} = \mathbf{U}_r\mathbf{U}_r^{\mathrm{T}}\mathbf{x} = \mathbf{P}\mathbf{x}$

Subspace reduction (➔ nodes in non-conductive / linear domain (2))

with $\mathbf{U}_r = \begin{bmatrix} \mathbf{U}_{1,r} \\ \mathbf{U}_{2,r} \end{bmatrix}$ subspace projector ➔ $\mathbf{P} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_{2,r}\mathbf{U}_{2,r}^{\mathrm{T}} \end{bmatrix}$ e.g. $\begin{bmatrix} \mathbf{x}_1 \\ \hat{\mathbf{x}}_2 \end{bmatrix} = \mathbf{P}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$

$$\hat{\mathbf{x}}_2 = \mathbf{U}_{2,r}\mathbf{z}_2$$

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Model Order Reduction

Subspace Reduction 2

Subdomain reduction for the EQS formulation

$\dim \mathbf{z}_2 \ll \dim \phi_2$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12}\mathbf{U}_{2,r}^T \\ \mathbf{U}_{2,r}^T\mathbf{M}_{21} & \mathbf{U}_{2,r}^T\mathbf{M}_{22}\mathbf{U}_{2,r}^T \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \phi_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\phi) & \mathbf{K}_{12}\mathbf{U}_{2,r}^T \\ \mathbf{U}_{2,r}^T\mathbf{K}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{U}_{2,r}^T\mathbf{b}_2 \end{bmatrix}$$

Matrix assembly / projection only once before time loop!

Subdomain reduction for the MQS formulation

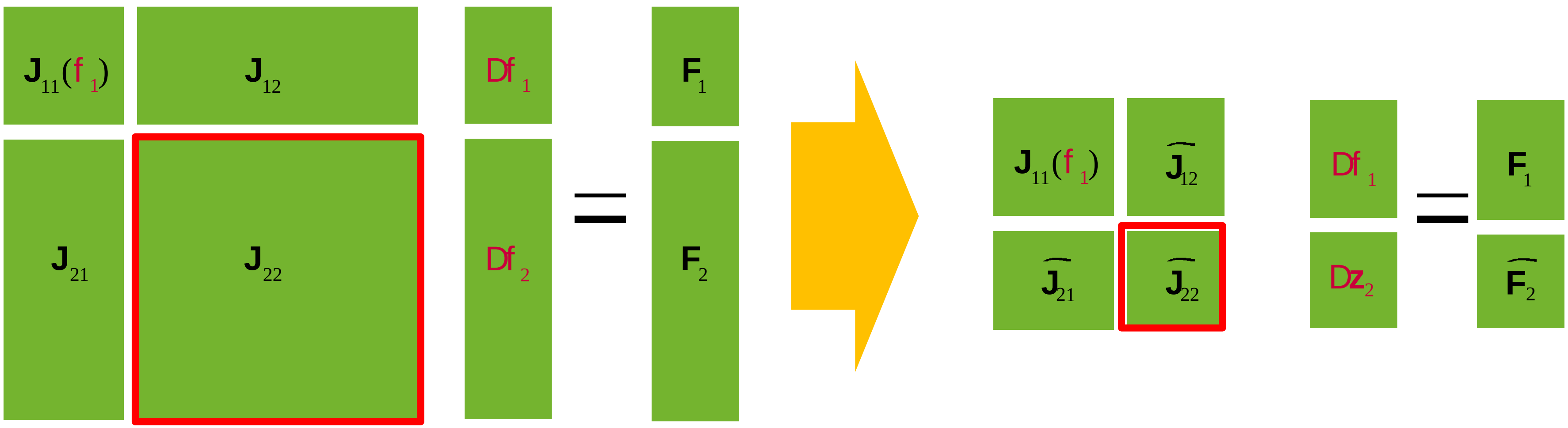
$\dim \mathbf{z}_2 \ll \dim \mathbf{a}_2$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\mathbf{a}) & \mathbf{K}_{12}\mathbf{U}_r \\ \mathbf{U}_r^T\mathbf{K}_{21} & \mathbf{U}_r^T\mathbf{K}_{22}\mathbf{U}_r \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{U}_r^T\mathbf{j}_2 \end{bmatrix}$$

with $\begin{bmatrix} \mathbf{a}_1 \\ \hat{\mathbf{a}}_2 \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix}$ and $\hat{\mathbf{a}}_2 = \mathbf{U}_{2,r}\mathbf{z}_2$

Use POD…

…, but only for the constant matrix subproblem



Subdomain reduction for the EQS formulation (MQS: analogously):

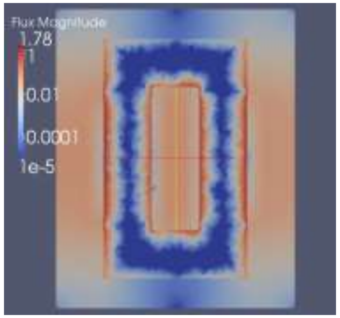
$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12}\mathbf{U}_{2,r}^{T} \\ \mathbf{U}_{2,r}^{T}\mathbf{M}_{21} & \mathbf{U}_{2,r}^{T}\mathbf{M}_{22}\mathbf{U}_{2,r} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \phi_1 \\ \mathbf{z}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\phi_1) & \mathbf{K}_{12}\mathbf{U}_{2,r}^{T} \\ \mathbf{U}_{2,r}^{T}\mathbf{K}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \mathbf{z}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{U}_{2,r}^{T}\mathbf{b}_2 \end{bmatrix}$$

where $\phi_2 \cong \mathbf{U}_{2,r}\mathbf{z}_2$ and $\dim \mathbf{z}_2 \ll \dim \phi_2$

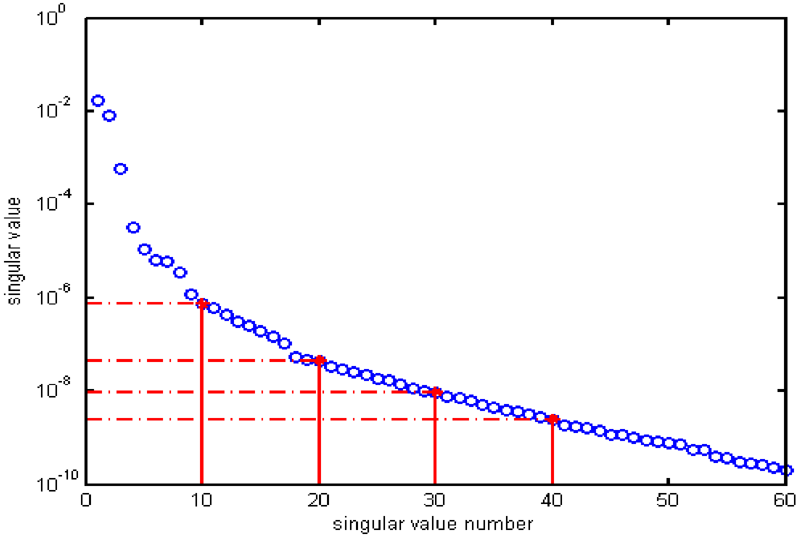
MQS Example

Transformer Model:

7713 nodes:
3363 in linear
subdomain

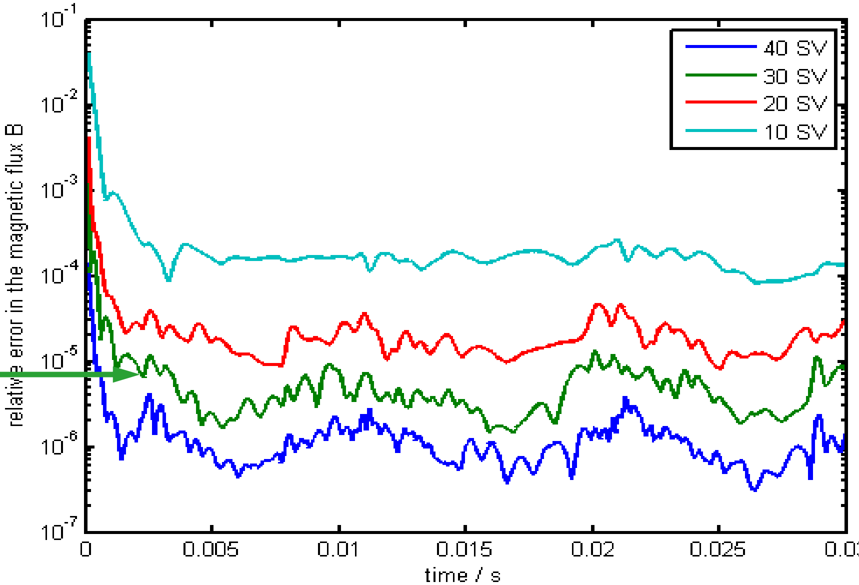


Transient sinus
excitation 50 Hz



Factor 100

Subspace reduction leads to
acceptable results in
magnetic flux



Speed-Up:

MQS/MOR ~1.6

➔ Here: Solution process is
dominated by the nonlinear effects

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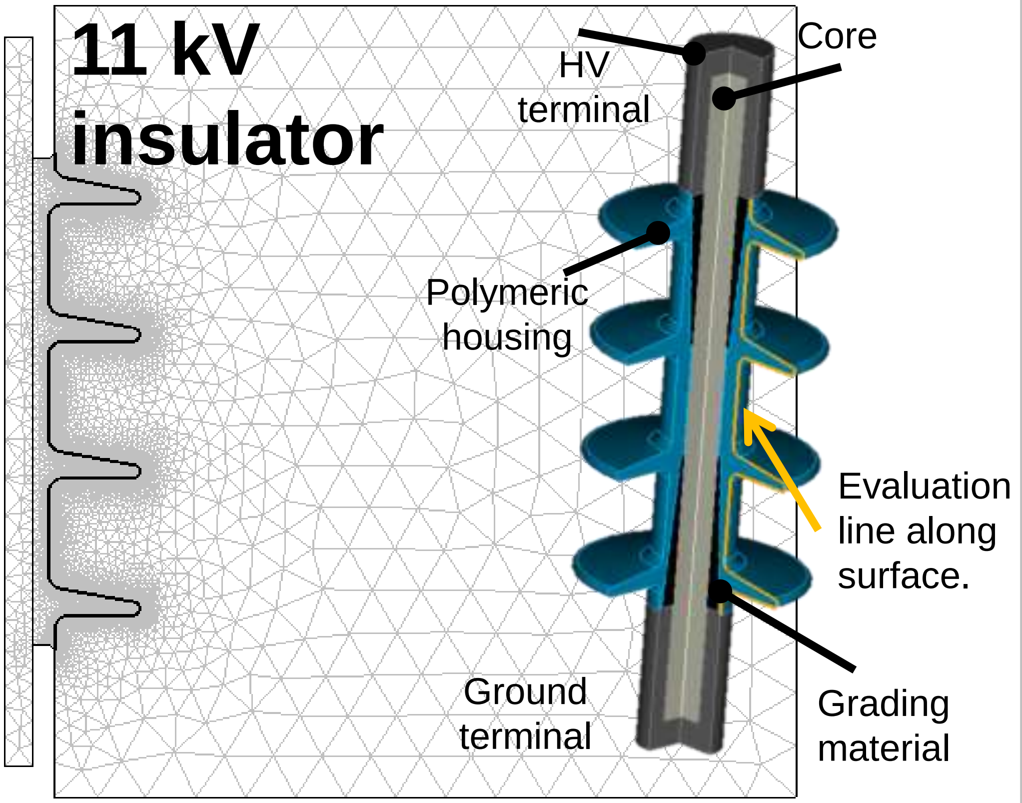
Challenges

- A single simulation is computationally heavy
- Uncertainty Quantification requires thousands of simulations
- but: only a few nonlinear degrees of freedom

→ MOR



R. Abd-Rahman, A. Haddad, N. Harid, and H. Griffiths. "Stress control on polymeric outdoor insu- lators using Zinc oxide microvaristor composites". In: IEEE Transactions on Dielectrics and Electrical Insulation 19.2 (2012), pp. 705–713. issn: 1070-9878.

doi: 10.1109/TDEI.2012.6180266.

Uncertainty Quantification

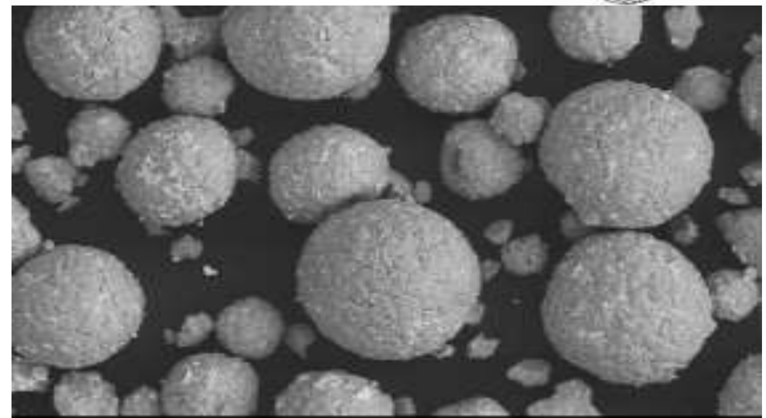
Material uncertainties:

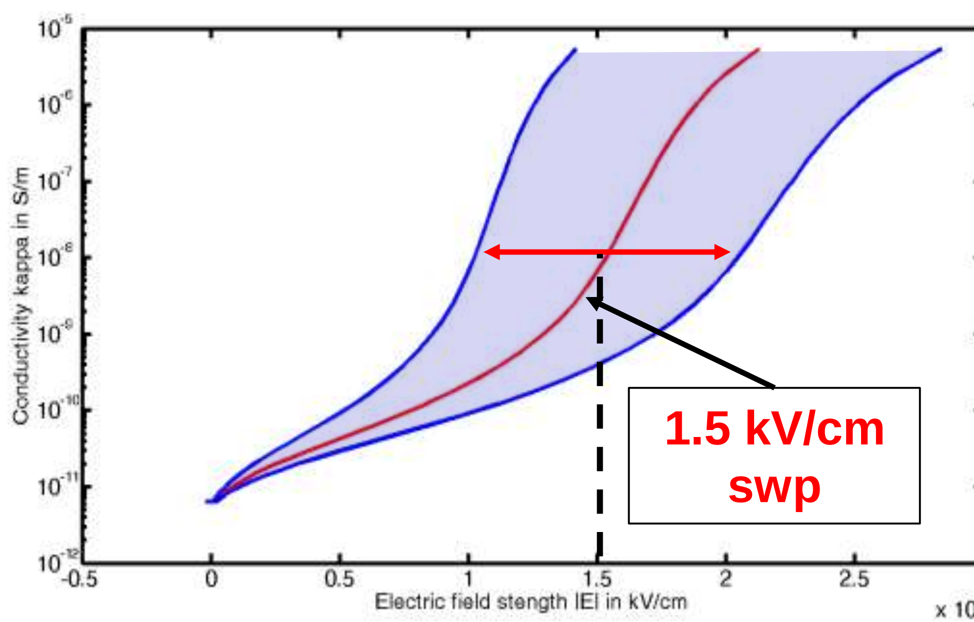
- Microvaristors are polymeric compounds with nonlinear Zinc Oxide (ZnO) fillers.
- Switching point determined by filler concentration.

Switching point (swp) variation due to production process.

Question:
How is the field effected by the swp variation?

Uncertainty Quantification

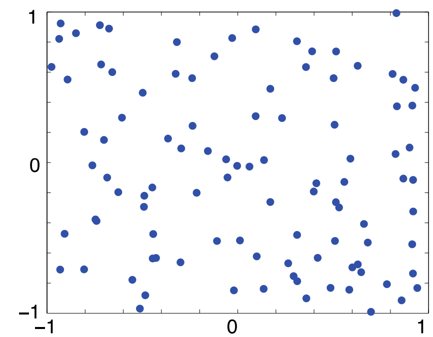

Microvaristors (ABB)



Mathematical Foundations

- Probability Space: (Ω, Σ, μ)
- Random Parameter (Switching Point): $E_{\mathrm{swp}}(\omega)$
- Random Process $\Phi(t,\omega) := \varphi\left(t, E_{\mathrm{swp}}(\omega)\right)$
- Expected Value: $\mathbb{E}\left(\Phi(t,\omega)\right) \approx \sum_k w_k \varphi(t, E_{\mathrm{swp}}^{(k)})$
- Variance: $\mathrm{var}\left(\Phi(t,\omega)\right) \approx \sum_k w_k \varphi(t, E_{\mathrm{swp}}^{(k)})^2 - \mathbb{E}\left(\Phi(t,\omega)\right)^2$

Quadrature vs. Monte Carlo



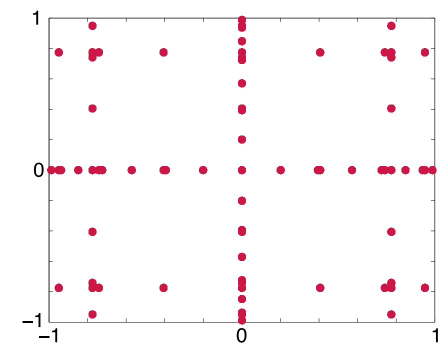
■ Stochastic Approach: Monte Carlo

– large number n of samples, i.e., full simulations

– Convergence independent of number of variables

– each simulation has equal weight $w = 1/n$

– samples are stochastic, i.e., obtained by random generator



■ Deterministic Approch: Quadrature

– efficient for a small number of random variables

– curse of dimensionality

– problematic for many variables

– weights and samples are determined by quadrature rules e.g. Gauss-Hermite for normally distributed variables

Switching Points of Field are Random Variables

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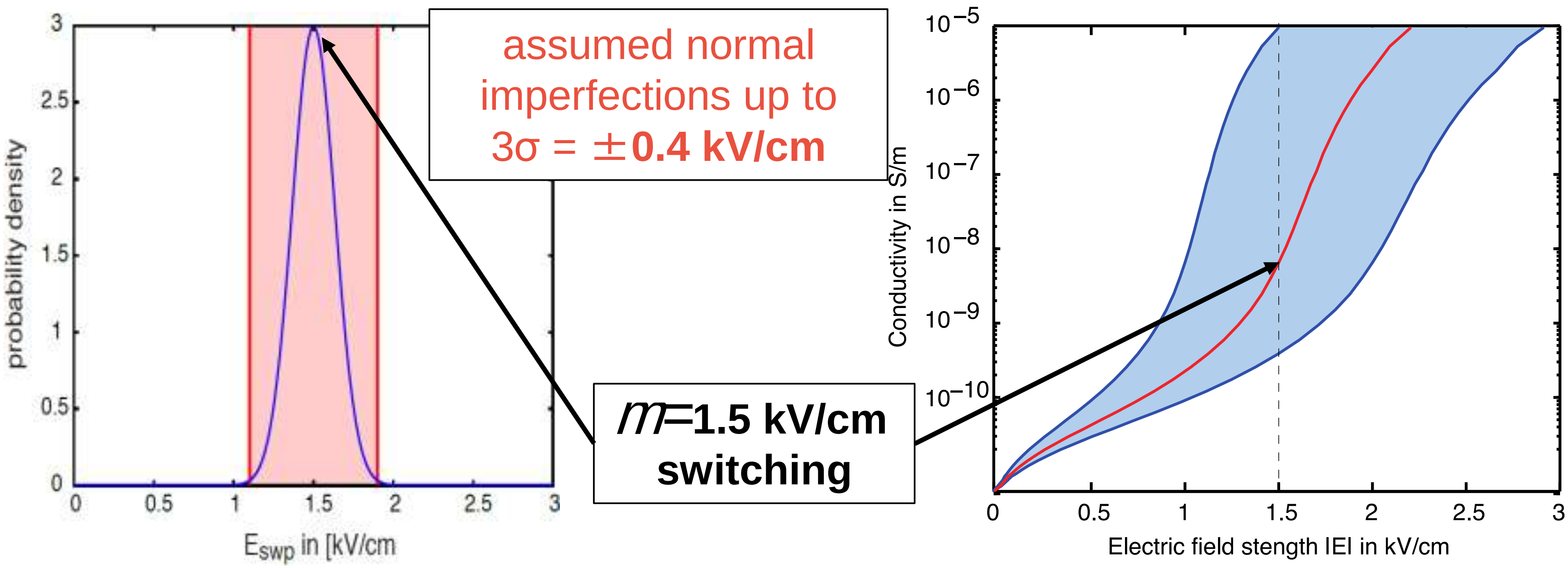
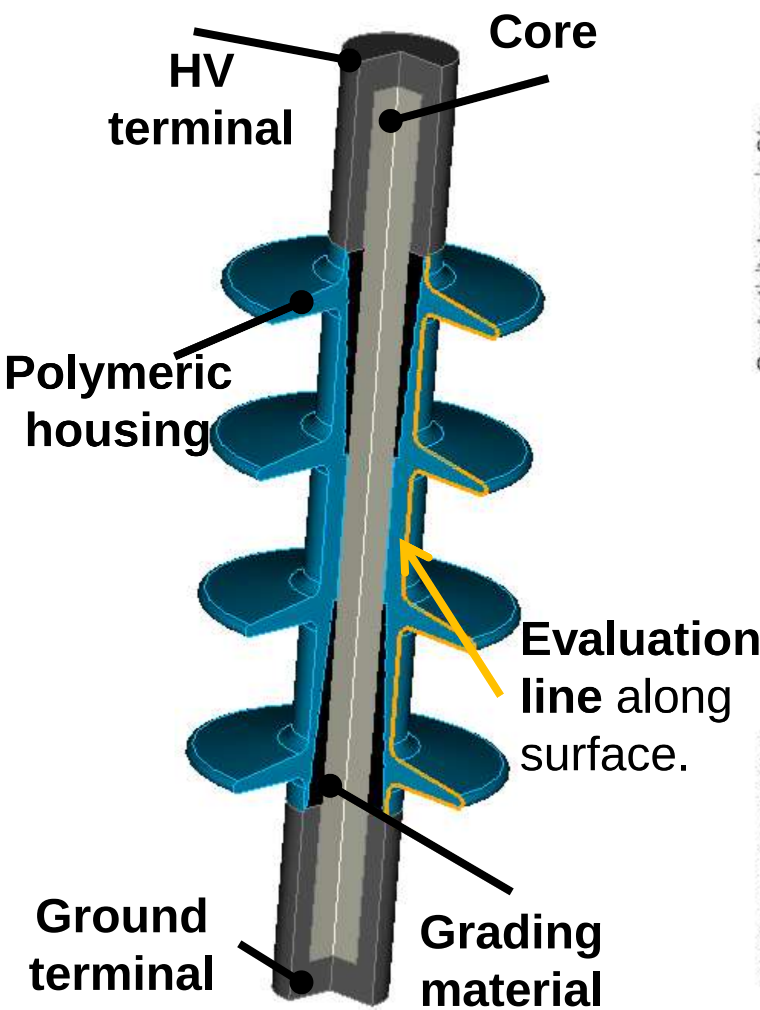
Input

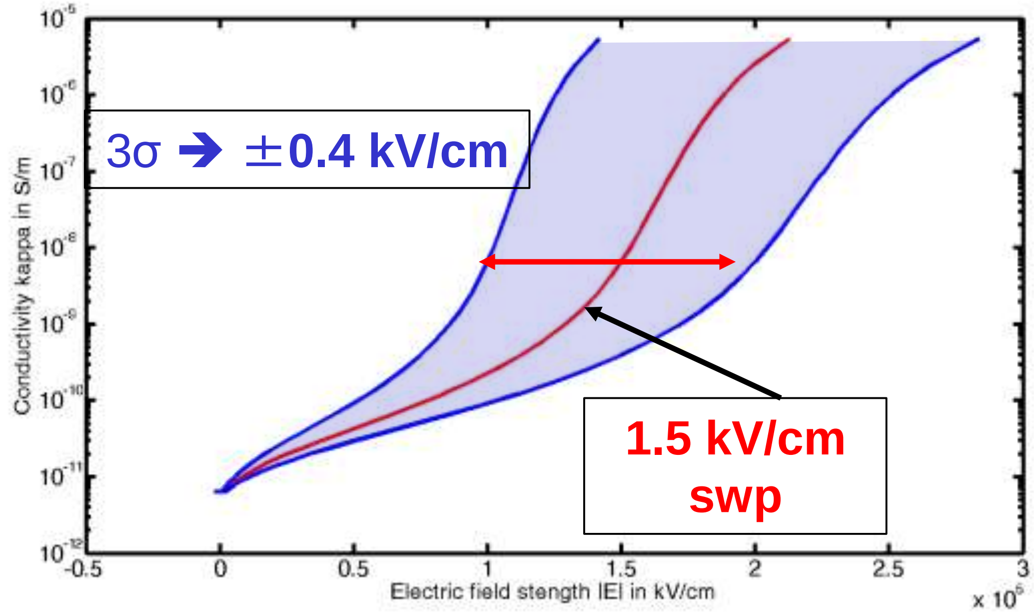
Assume that the switching point field strength is a random variable, i.e.,

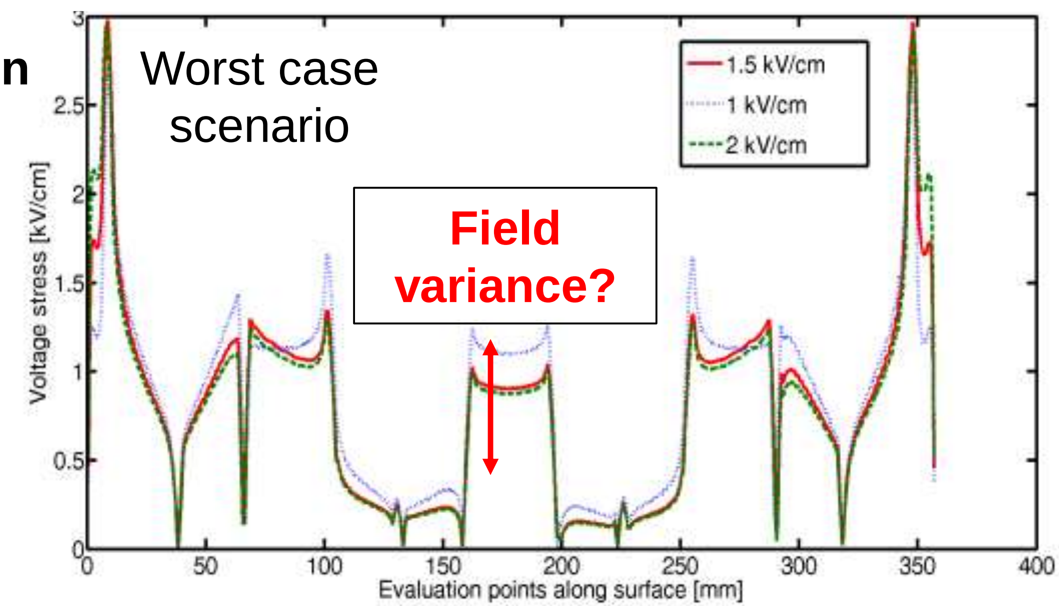
$$E_{swp} \sim N(\mu, \sigma^2)$$ *m* mean, *s* standard deviation



EQS Example

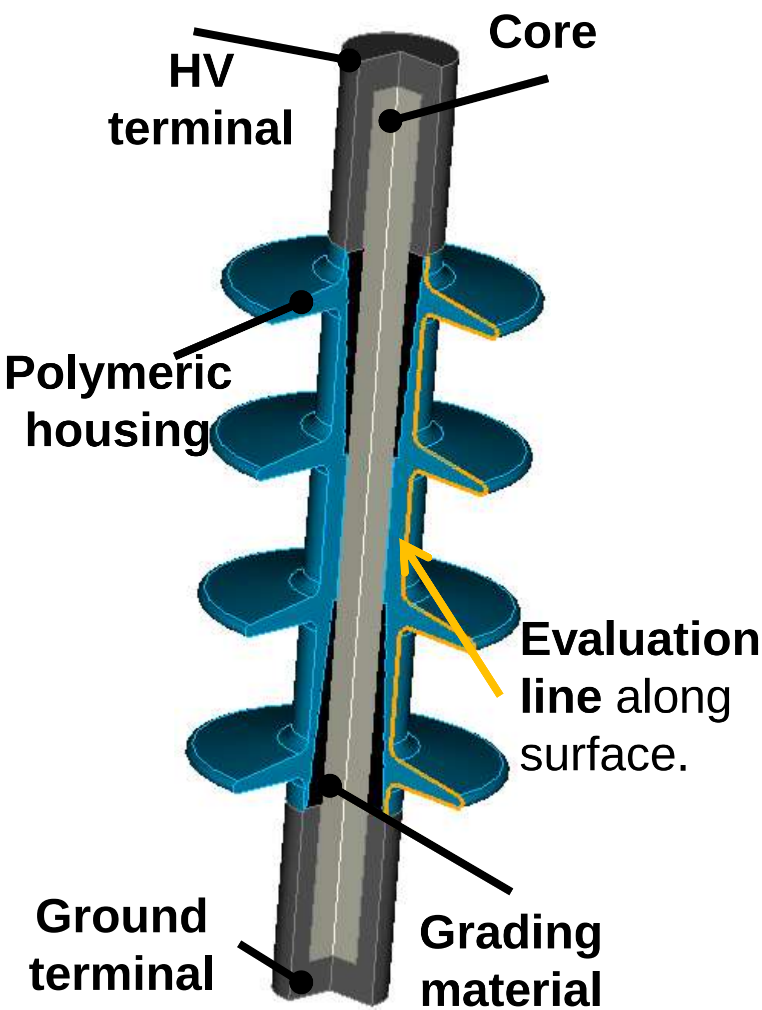






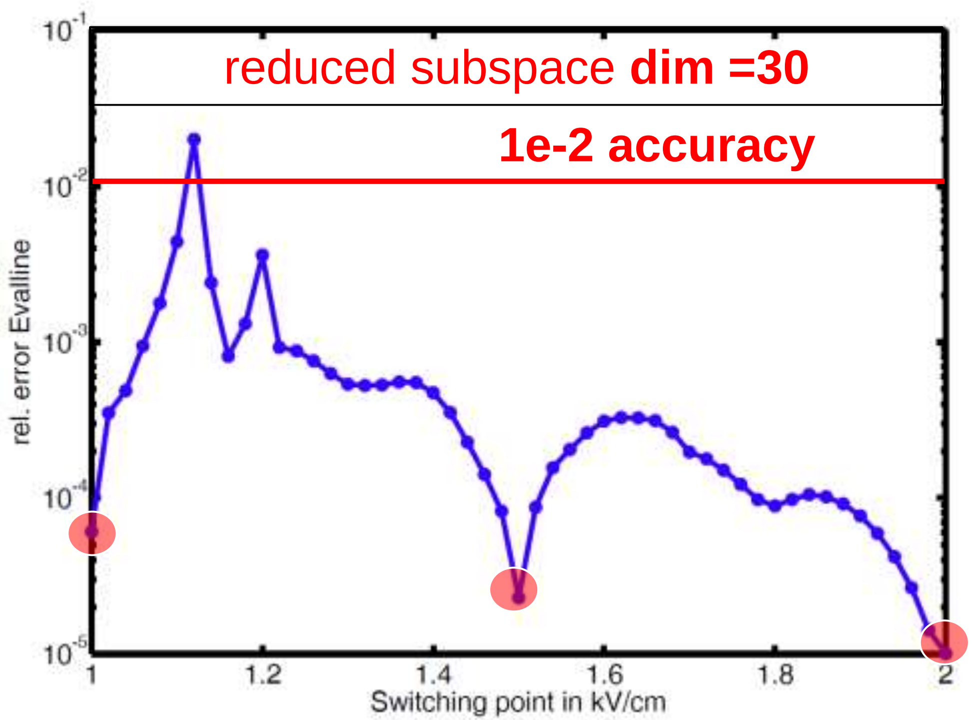
9880 Nodes
8790 in linear subspace

EQS Example



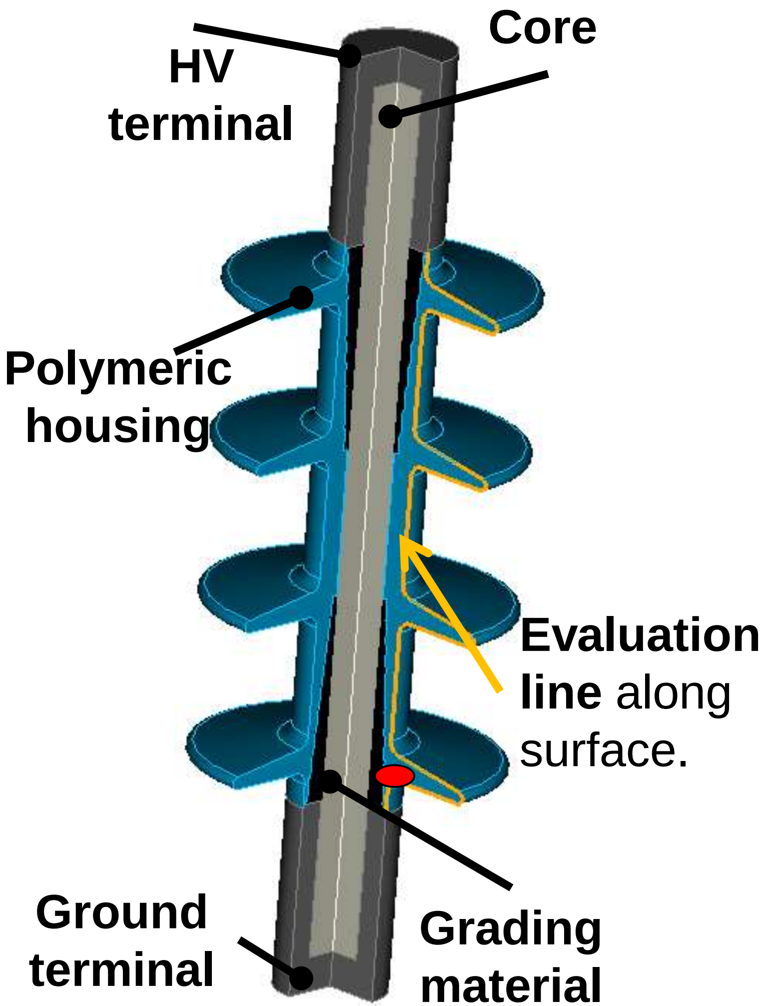
How to build the snapshot matrix?

- Max. & min. values random generator issued, e.g. 1 kV/cm and 2 kV/cm
- Further full simulation at 1.5 kV/cm



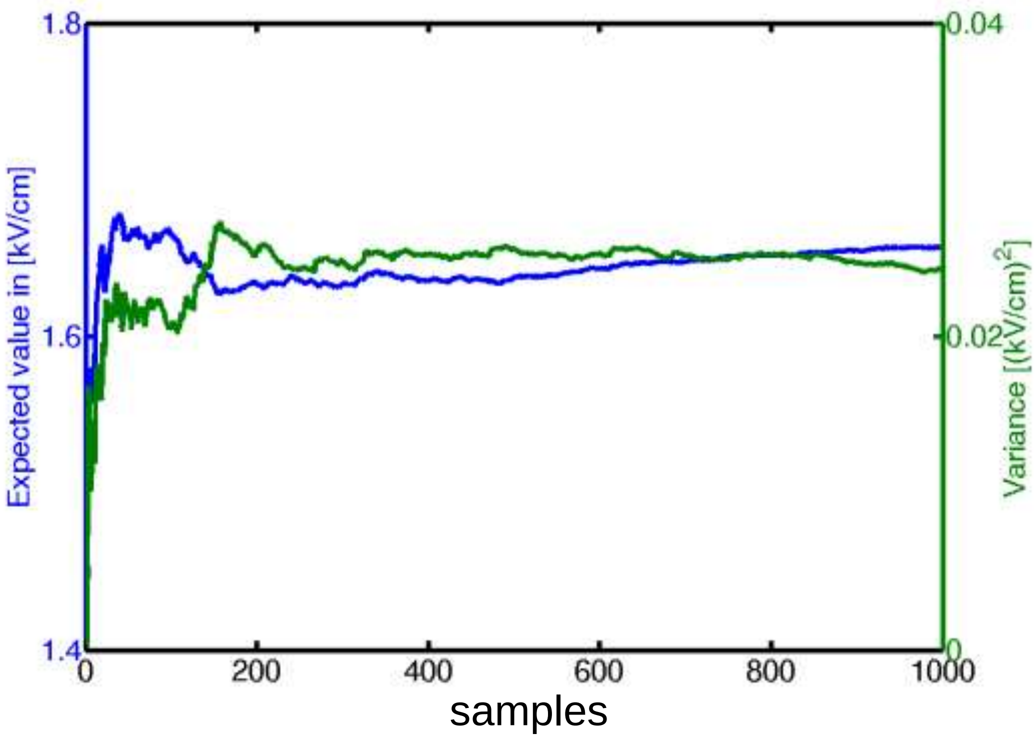
- ➔ We add further full simulation at 1.12 kV/cm to further improve accuracy

EQS Example



How many runs for Monte Carlo?

- Expected value and variance for evaluation point 8 (4 mm from below)



> 1000 TEQS simulations required !!!

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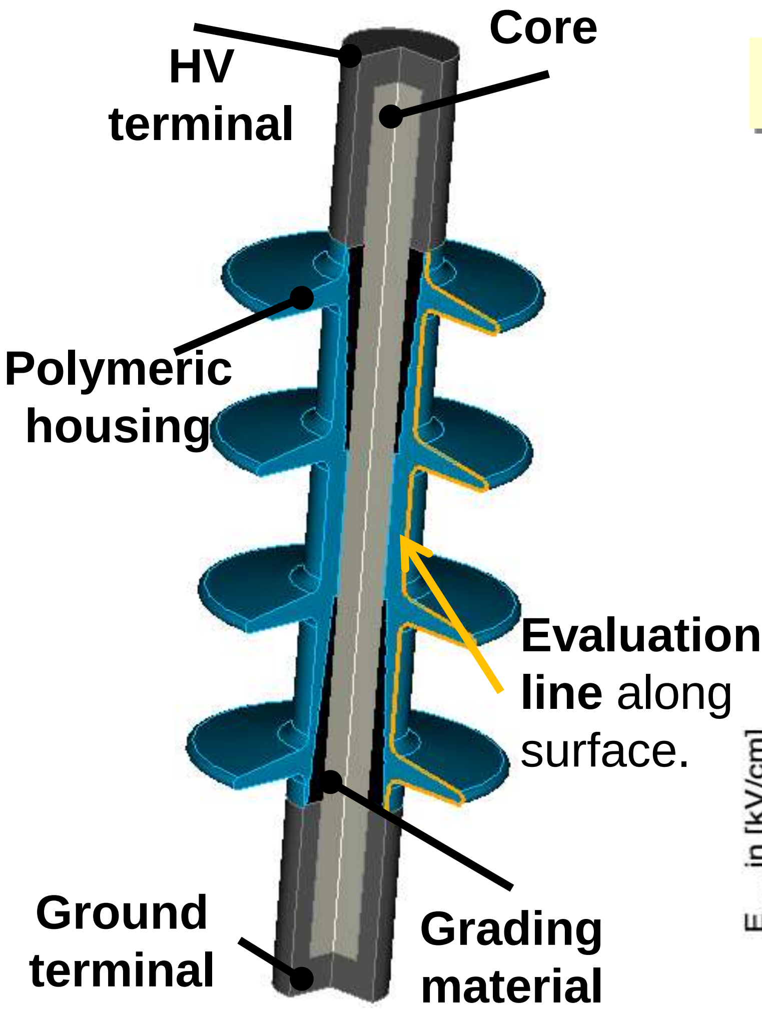
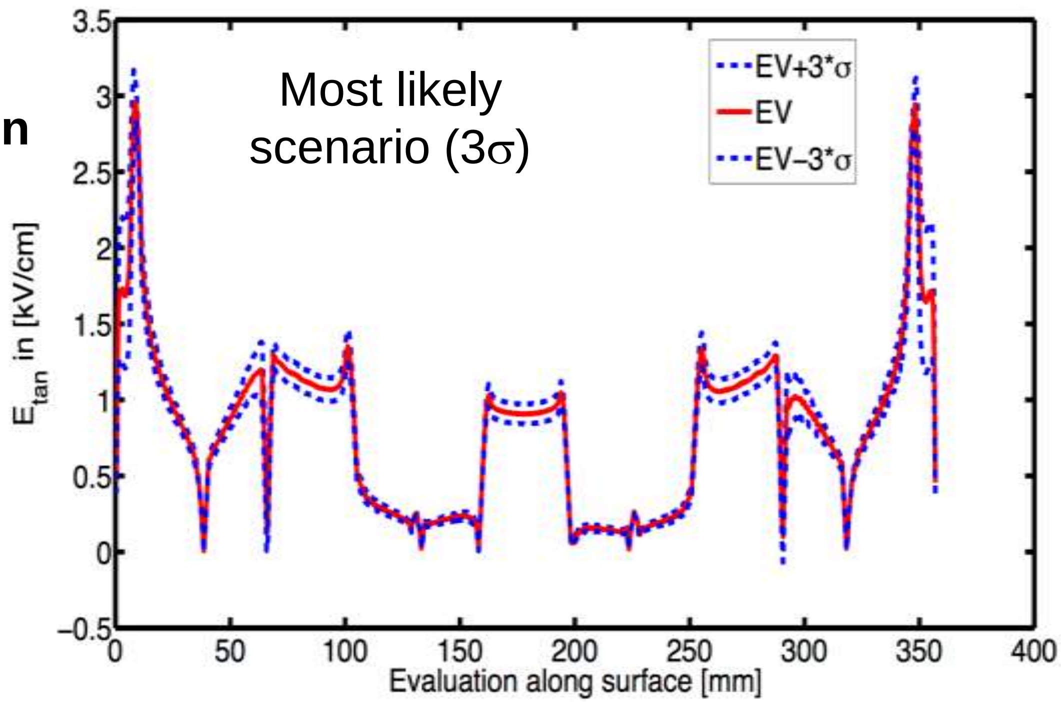
EQS-Example



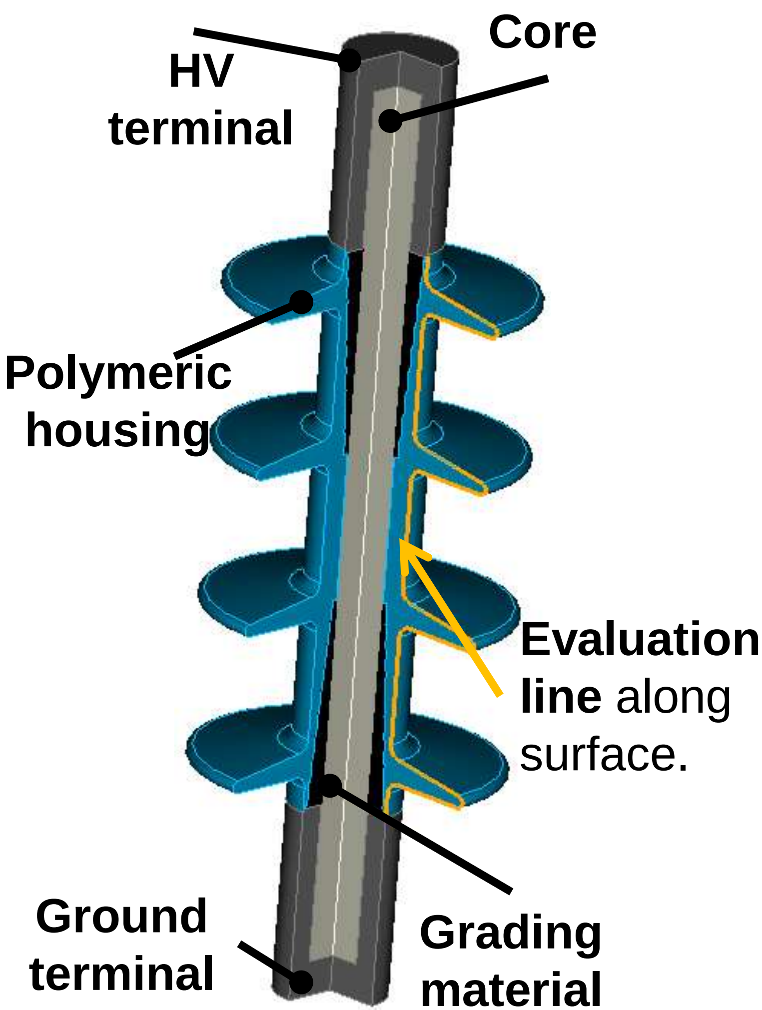
Results

- Snapshot build with full simulations to **four** switching points
  ➔ dimension of reduced space: **30**
- MOR-sim. time*: **167 hours** (1-CPU-Core)
- Rel. Error of Expected Value < **1e-3**
- Rel. Error of Variance < **5.4e-2**

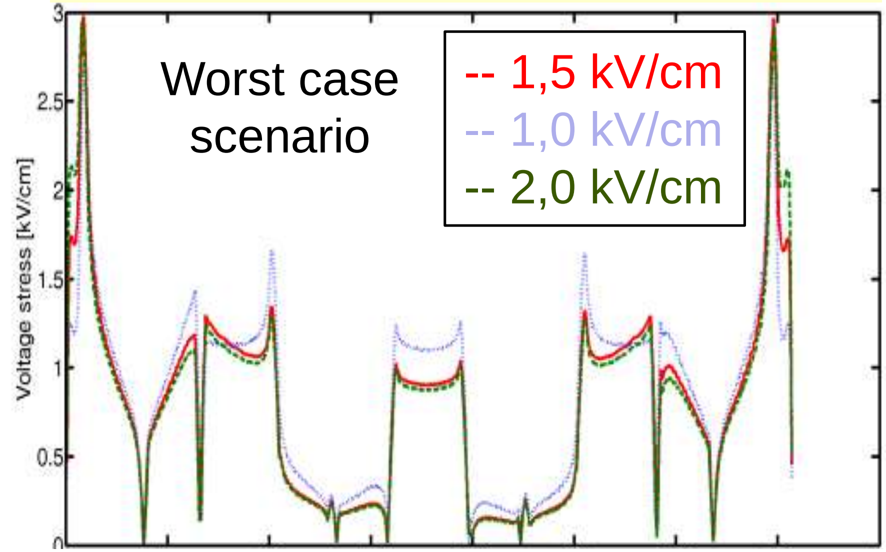


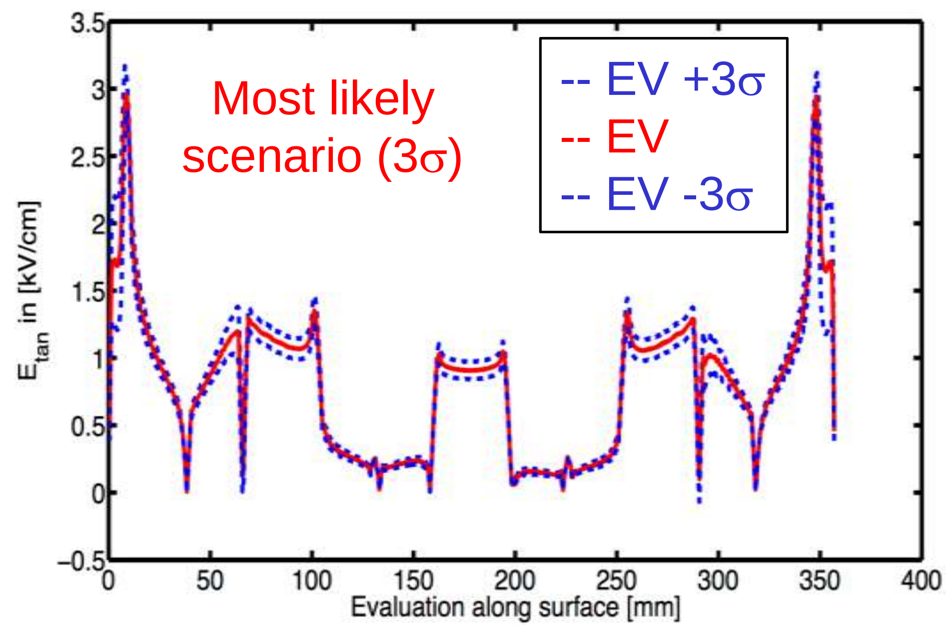
*Uncertainty quantification full simulations ~250 h

EQS Example



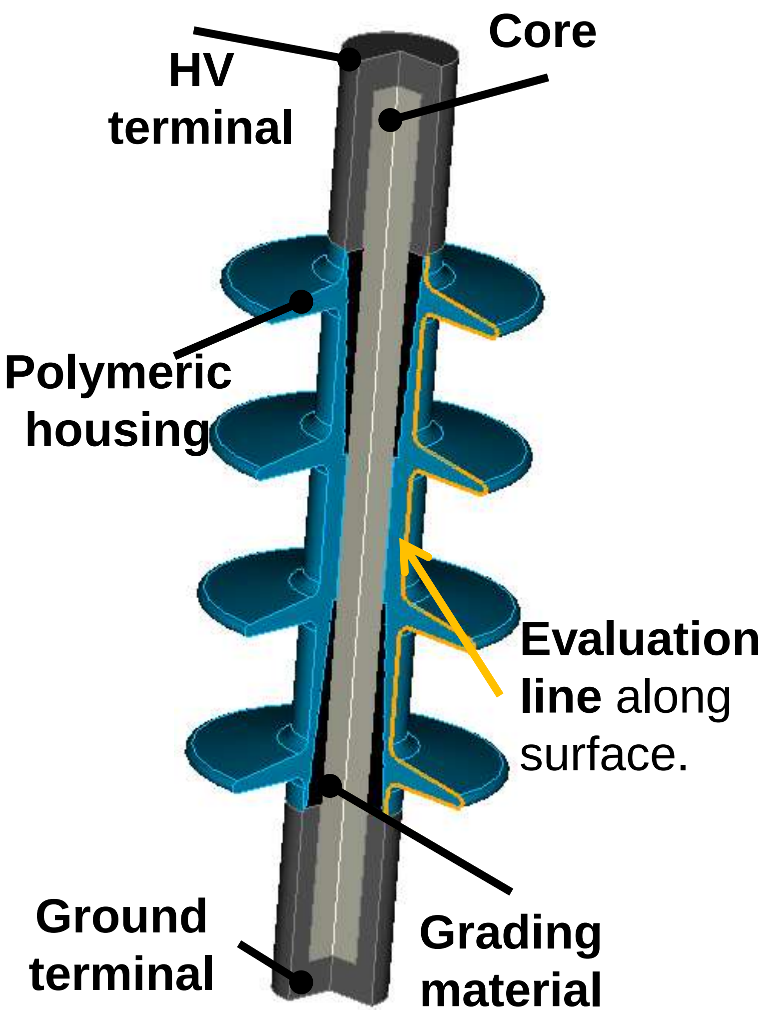
Results: Field Variance

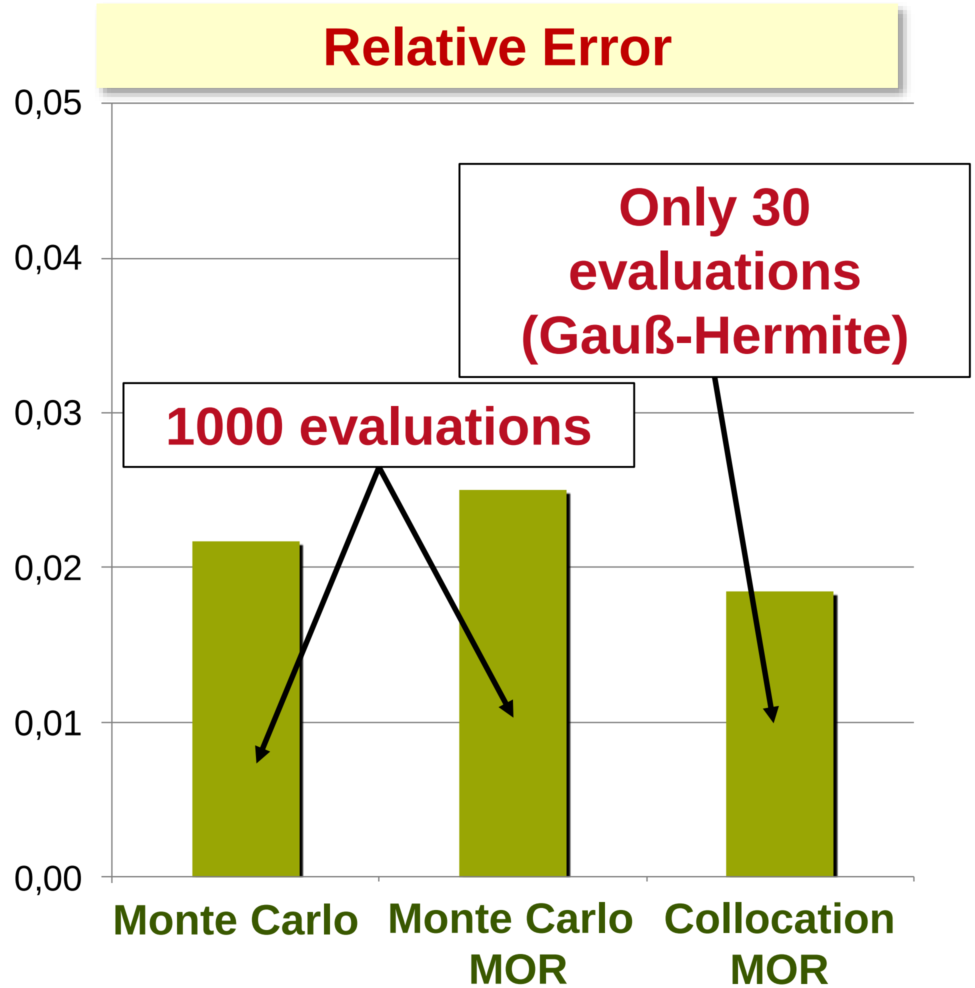




*Uncertainty quantification full simulations ~250 h

Accuracy of the Results in the Uncertainty Quantification





Compared with high accuracy quadrature

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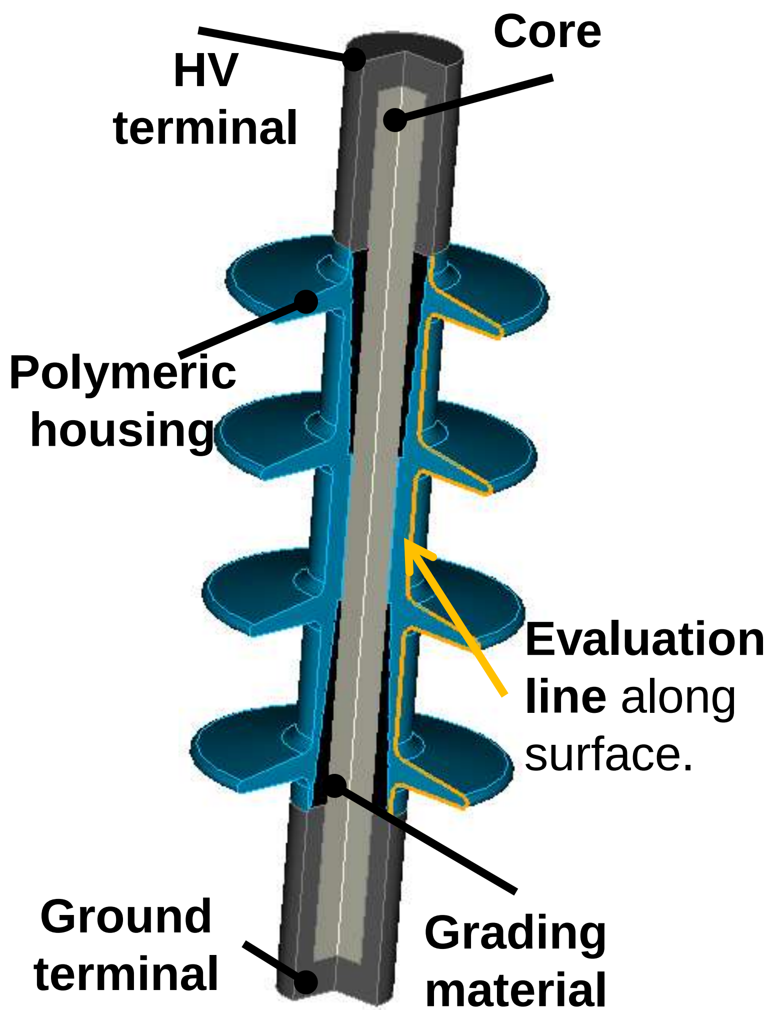
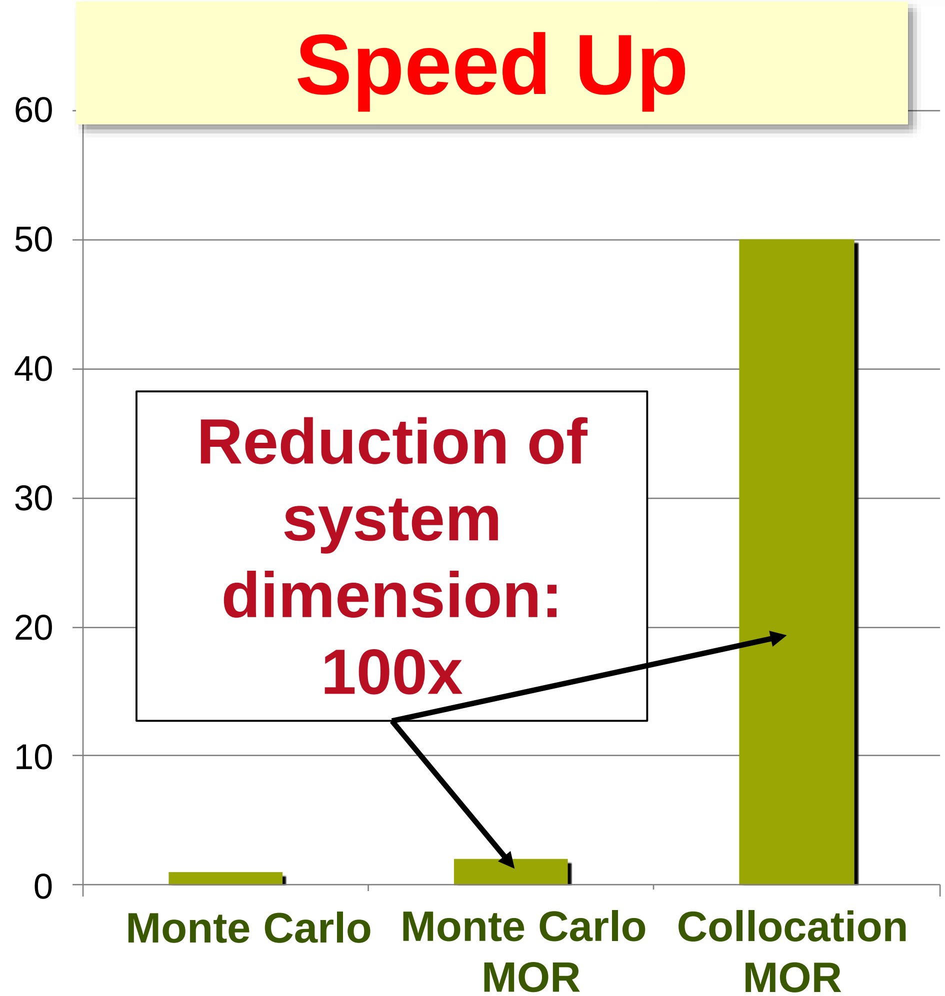
Speed-Up of the Computation in the Uncertainty Quantification





Snapshot build with full simulations of **four** switching points (**number of SVD is 30)**

Summary

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Efficient (non-)**linear** reduction technique

- Only a **few non-linear degrees of freedom** determine the uncertainty
- **Model order reduction** of the large linear subdomain

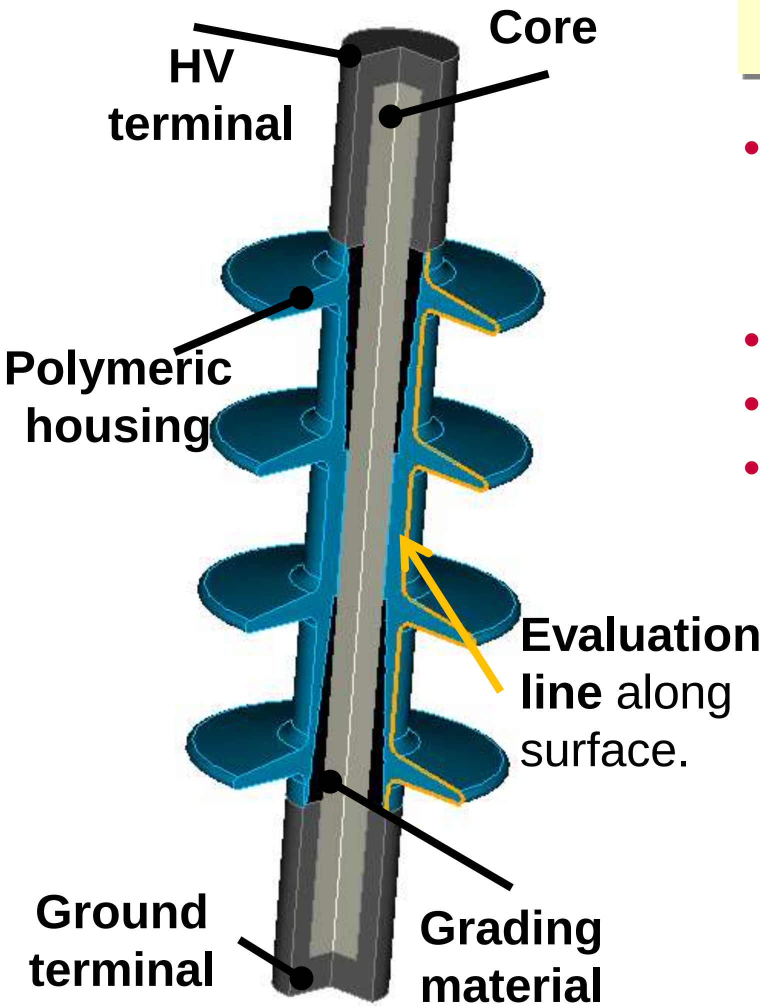
Application of linear subspace reduction: **Uncertainty Quantification**

- **Monte Carlo:** >1000 parameter-dependent nonlinear TEQS simulations required
- **Stochastic Collocation**: only 30 evaluations necessary (for only one variable, but: curse of dimensionality!)

Thank you
for your attention!

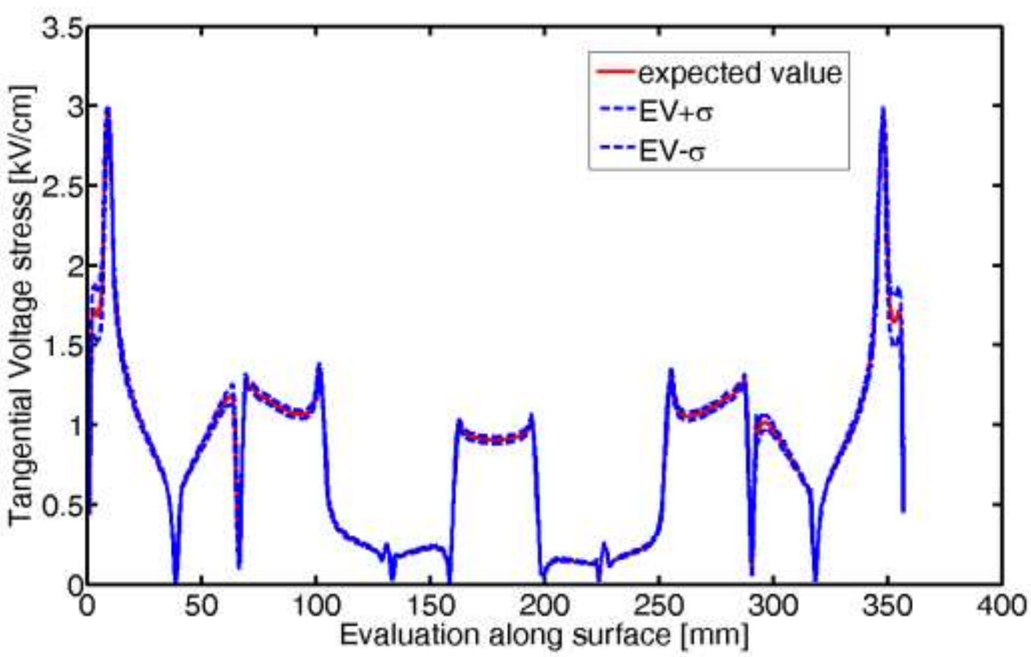
Contents

EQS-Example



Results

- Snapshot build with full simulations to **four** switching points
  ➔ dimension of reduced space: **30**
- MOR-sim. Time* **167 hours** (1-CPU-Core)
- Rel. Error of Expected Value < **1e-3**
- Rel. Error of Variance < **5.4e-2**



*Uncertainty quantification full simulations ~250 h

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Outline

- **Motivation**
- **Quasistatic Fields**
  - –Slowly varying Fields
  - –Quasistatic Field Approximation
  - –System Sub-structuring
- **Linear Subproblem Model Order Reduction**
  - –Proper Orthogonal Decomposition
- **Uncertainty Quantification using MOR**
  - –Stochastic Setup
  - –Electro-Quasistatic Example
- **Summary**

BERGISCHE UNIVERSITÄT WUPPERTAL

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Outline

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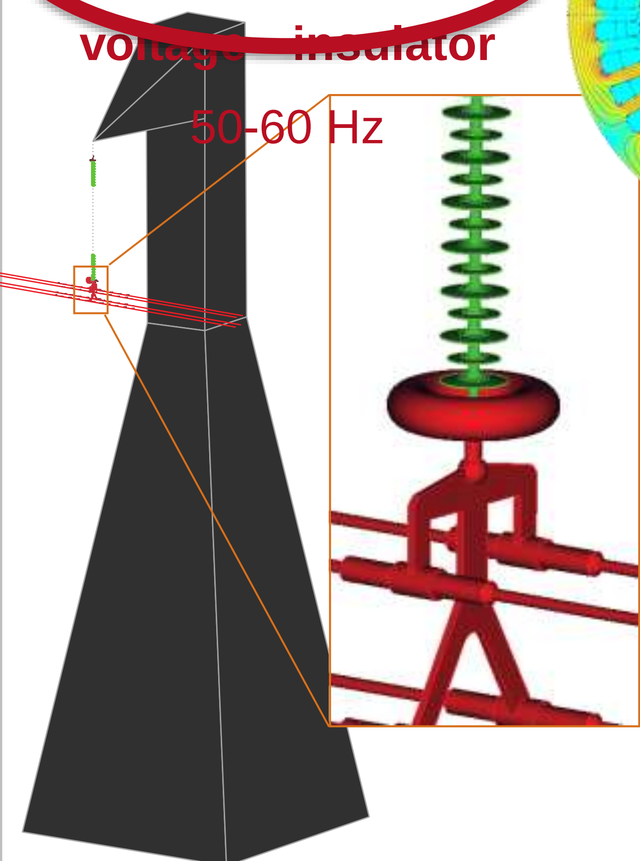
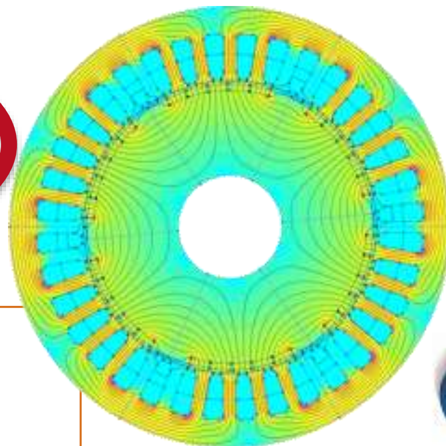
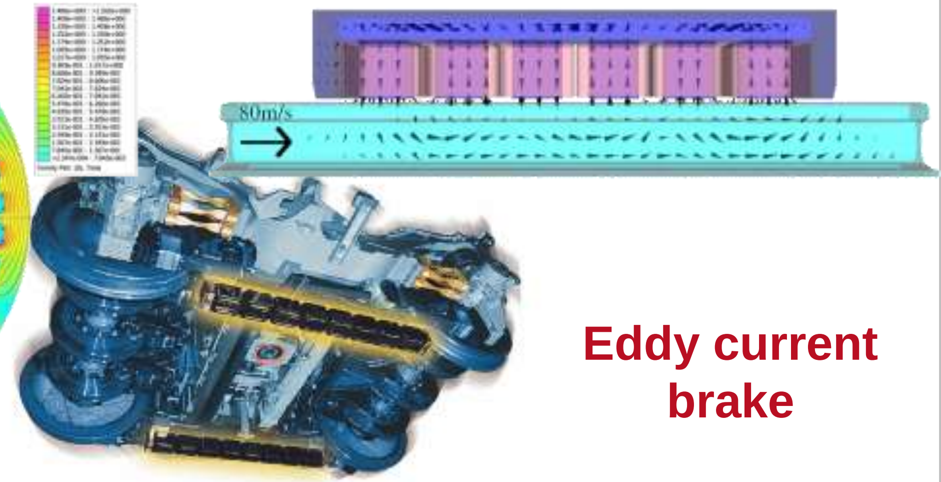
Motivation: Slowly Varying Electromagnetic Fields

High voltage insulator
50-60 Hz

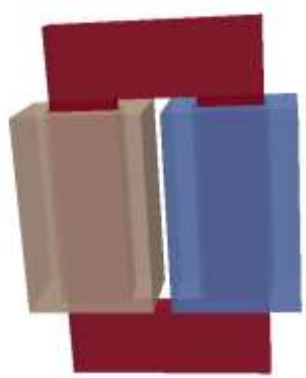


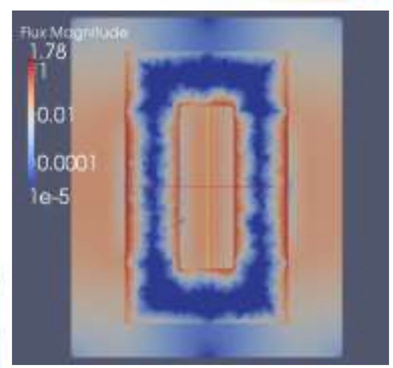


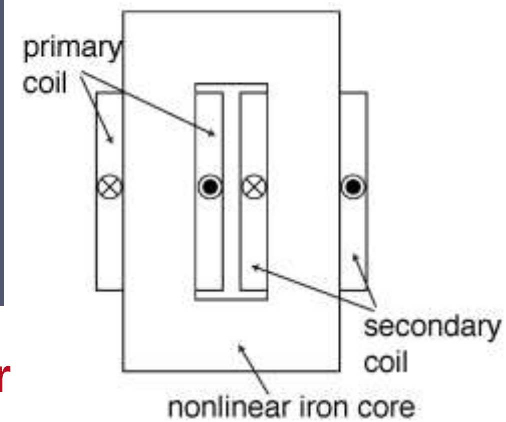
Electrical machines
50-60 Hz



Eddy current brake







Transformer
50-60 Hz

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Outline

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  - –Slowly varying Fields
  - –Quasistatic Field Approximation
  - –System Sub-structuring
- **Linear Subproblem Model Order Reduction**
  - –Proper Orthogonal Decomposition
- **Uncertainty Quantification using MOR**
  - –Stochastic Setup
  - –Electro-Quasistatic Example
- **Summary**

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Quasistatic Field Approximation

Slowly Varying Field Approximations

Neglecting inductive effects:	Neglecting capacitive effects:
$\cancel{\frac{\P \vec{B}}{\P t}}$ **Electro-Quasistatic EQS**	$\cancel{\frac{\P \vec{D}}{\P t}}$ **Magneto-Quasistatic MQS**

Applied to **Maxwell** equations

Electric scalar potential formulation	Magnetic vector potential formulation
$\frac{\partial}{\partial t}\nabla\cdot(e\nabla f(t))+\nabla\cdot(k(f)\nabla f(t))=0$	$k\frac{\partial}{\partial t}\vec{A}(t)+\nabla\times(n(\vec{A})\nabla\times\vec{A}(t))=\vec{J}_s(t)$

Initial-boundary value problems

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State of the Art

Applying spatial discretizations (FEM, FIT,…)

Electro-Quasistatic	**Magneto-Quasistatic**
$\mathbf{M}_e \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{f} + \mathbf{K}_k(\mathrm{f})\mathrm{f} = rhs$	$\mathbf{M}_k \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} + \mathbf{K}_n(\mathbf{a})\mathbf{a} = \mathbf{j}_s$

State-of-the-Art: (linear) Model Order Reduction for Quasistatics:

- **„Brute Force Approach"** ➔ **Faster Solvers:**
  Many-core- /GPU-computing,
  better alg. solvers/preconditioners,
  novel formulations
- **„Condensate the Problem"** ➔ Equivalent circuit models,
  Model Order Reduction

State of the Art

Applying spatial discretizations (FEM, FIT,…)

Electro-Quasistatic

$$\mathbf{M}_e \frac{\mathrm{d}}{\mathrm{d}t}\varphi + \mathbf{K}_k(\varphi)\varphi = rhs$$

Magneto-Quasistatic

$$\mathbf{M}_k \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a} + \mathbf{K}_n(\mathbf{a})\mathbf{a} = \mathbf{j}_s$$

Only **non-linear** conductive part has relevant conductivity.

Only conductive part has **non-linear** reluctivity.

Split systems into **conductive (1)** and **non-conductive (2)** parts.

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\varphi_1) & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{11}(\mathbf{a}_1) & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{j}_1 \\ \mathbf{j}_2 \end{bmatrix}$$

Stiff non-linear ODE

DAE

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  - –Quasistatic Field Approximation
  - –System Sub-structuring
- **Linear Subproblem Model Order Reduction**
  - –Proper Orthogonal Decomposition
- **Uncertainty Quantification using MOR**
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Model Order Reduction: POD
Proper Orthogonal Decomposition

- Assemble system dynamics (solutions „snapshots“) $\mathbf{x}^{\alpha} = \mathbf{x}(t_{\alpha})$ in

  $\mathbf{X} = [\mathbf{x}^1, \ldots, \mathbf{x}^N]$ „**Snapshot matrix**“ of time step solutions

- Singular value decomposition (**SVD**)

  $$\mathbf{X} = \mathbf{U\Sigma V}^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_p \mathbf{u}_p \mathbf{v}_p^T \qquad \sigma_i \geq \sigma_{i+1} \geq 0$$

- Selecting the reduced basis (How many SVs should be selected?)

$$\mathrm{I}(r) = \frac{\sigma_1 + \cdots + \sigma_r}{\sigma_1 + \cdots + \sigma_r + \sigma_{r+1} + \cdots + \sigma_p} \approx \mathbf{1} \quad \Rightarrow \quad r$$

„Relative Information Criterion“

$$\left\| \mathbf{X} - \mathbf{X}_r \right\|_2 = \min_{rank(\mathbf{A}) \leq r} \left\| \mathbf{X} - \mathbf{A} \right\|_2 = \sigma_{r+1}(\mathbf{X})$$

„2-Norm“

with $\mathbf{X}_r = \mathbf{U}_r \mathbf{\Sigma V}_r^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$

„Low-Rank Approximation“

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  - –Slowly varying Fields
  - –Quasistatic Field Approximation
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Uncertainty Quatification

Material Uncertanties:

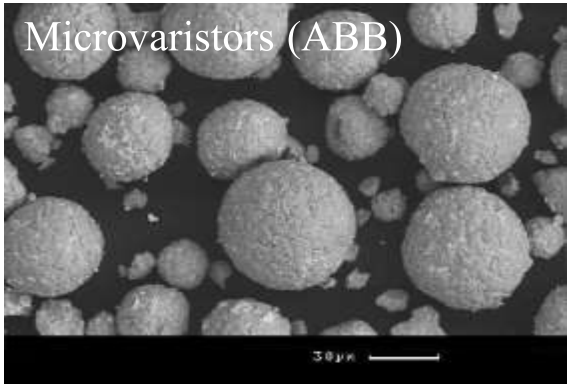
- Microvaristors are polymeric compounds with nonlinear Zinc Oxide (ZnO) fillers.
- Switching point determined by filler concentration.

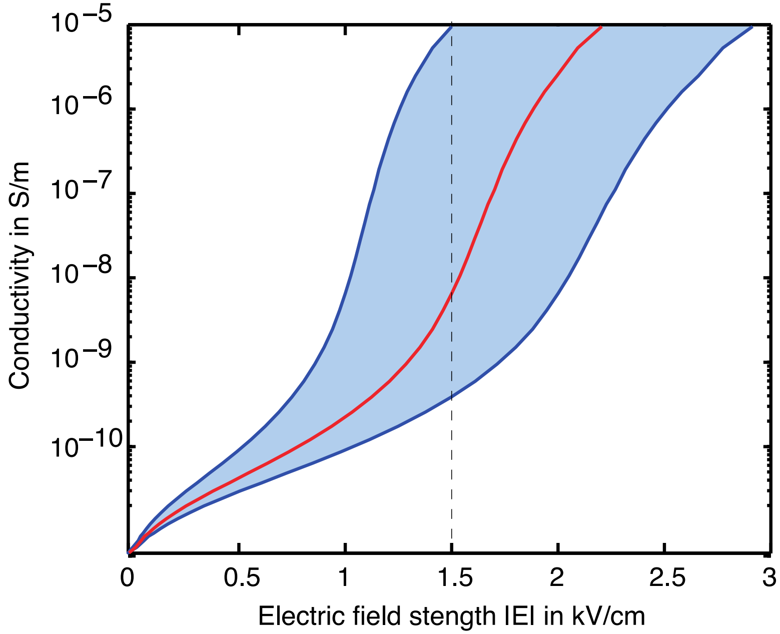
- Switching point (swp) variation due to production process.

Question:

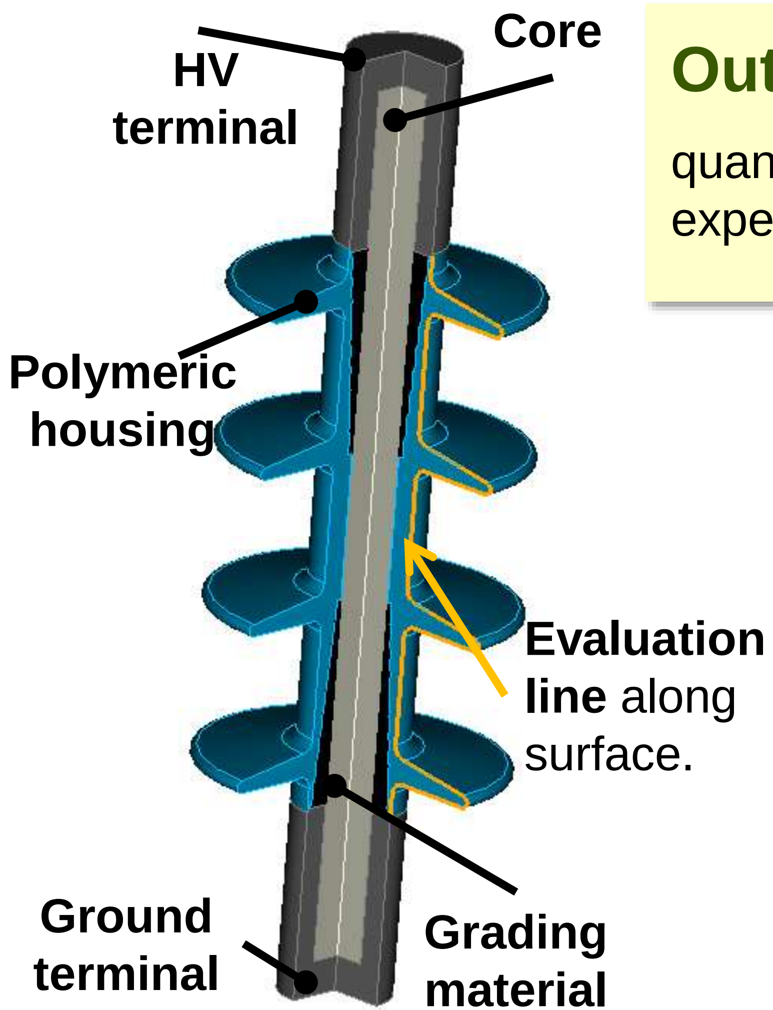
- How is the field effected by the variation of the switching point?

Uncertainty Quantification



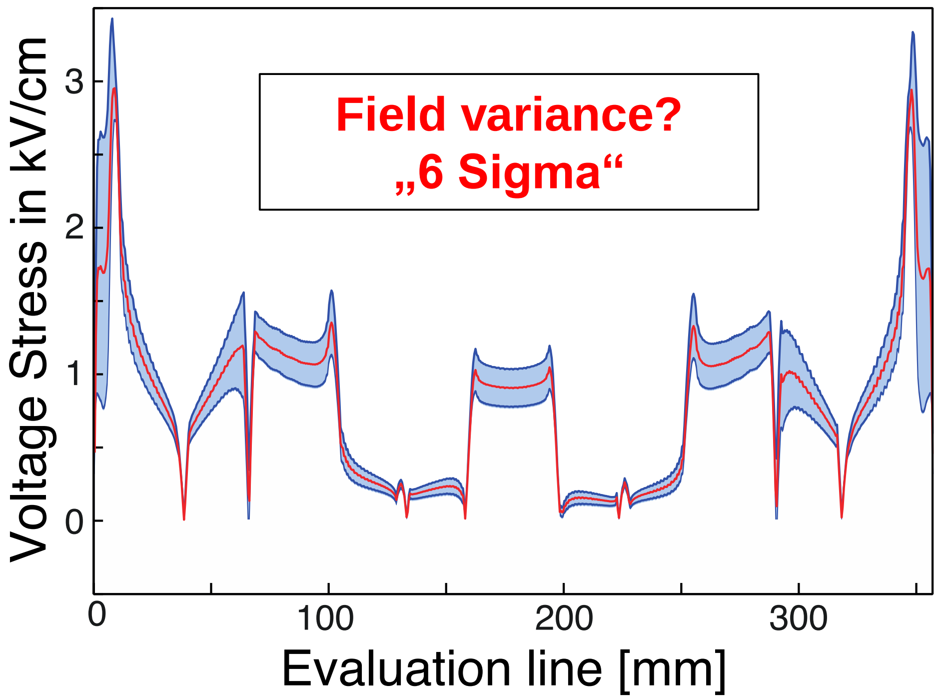


Solution is a Random Field

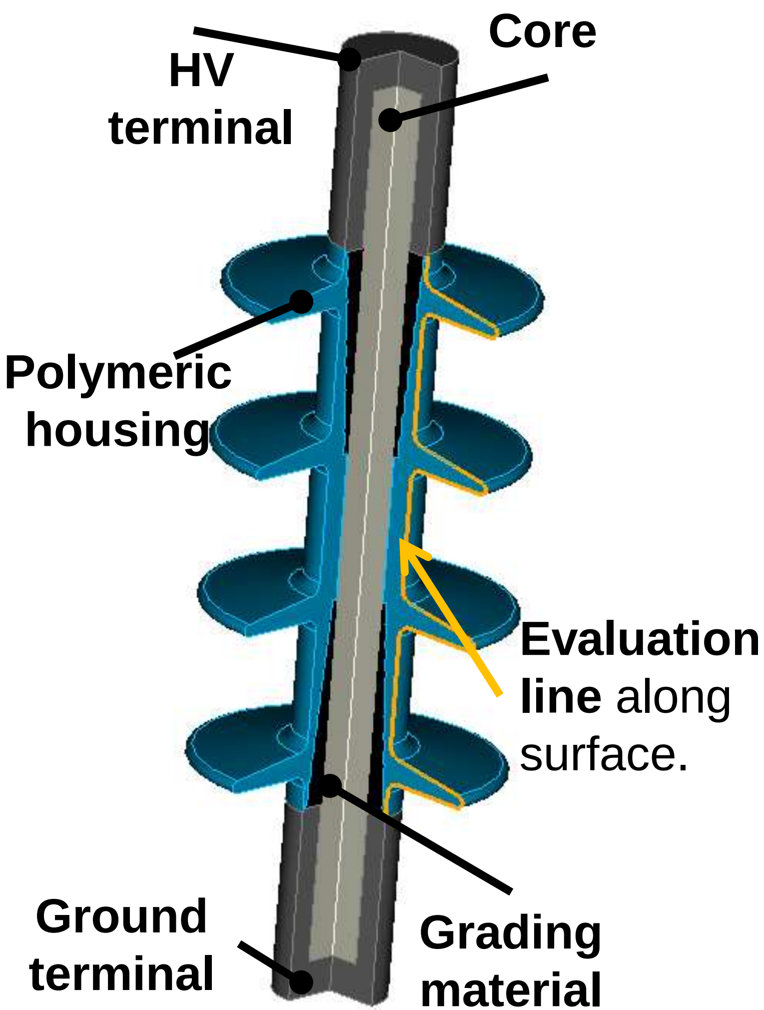


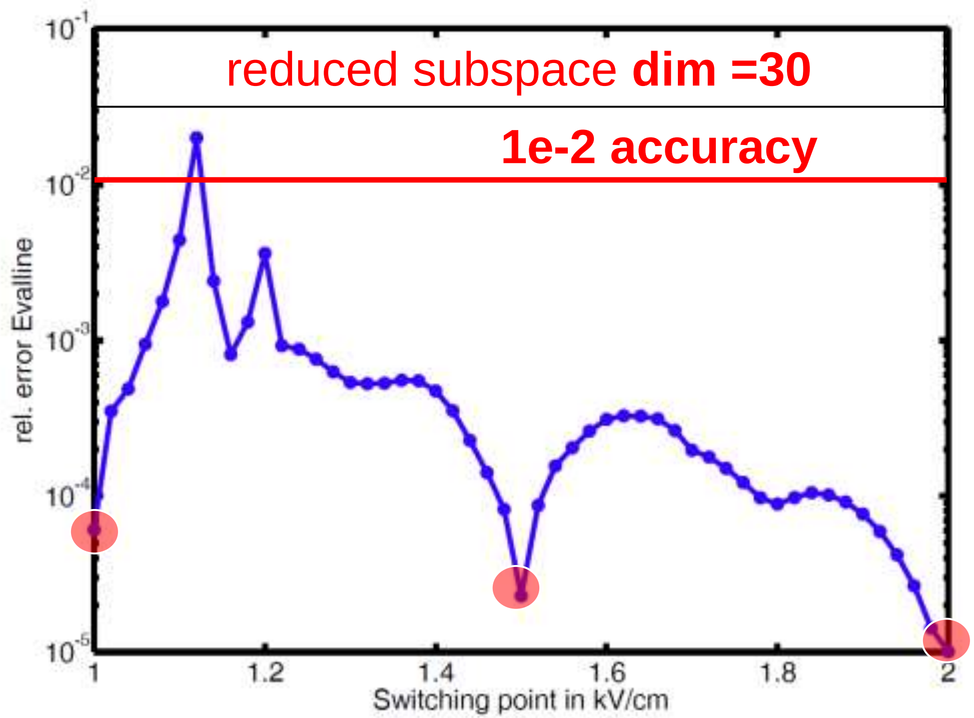
Output

quantify the effect to the electric field → calculate expected value (EV) & variance (VA)



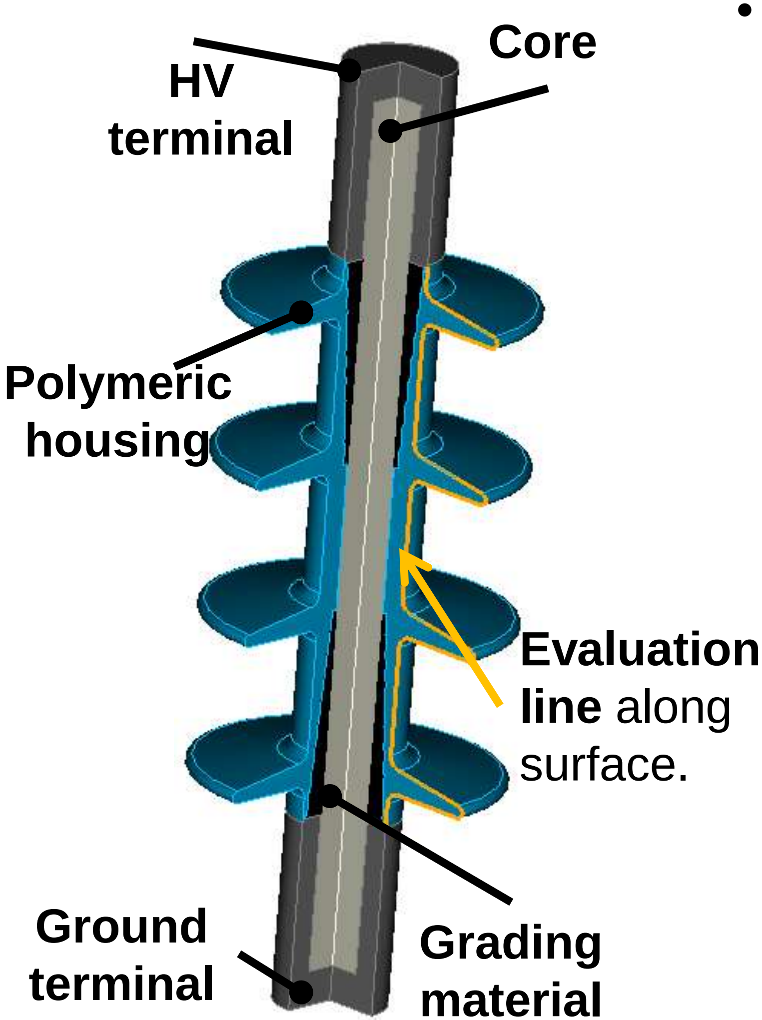
How to Build the Snapshot Matrix?



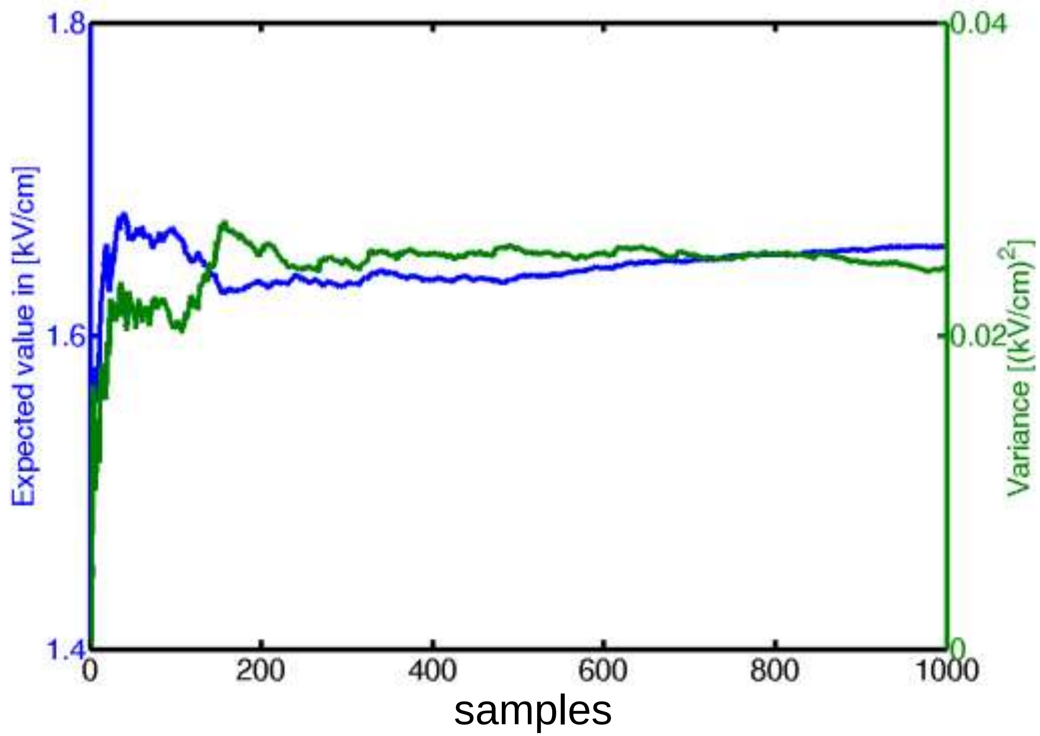


- Max. & min. values random generator issued, e.g. 1 kV/cm and 2 kV/cm
- Further full simulation at 1.5 kV/cm

How Many Runs for Monte Carlo?



- Expected value and variance for evaluation point 8 (4 mm from below)



> 1000 simulations required !

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Thank you for your attention!

- please ask questions now
- or write an email: schoeps@gsc.tu-darmstadt.de
- supported by the DFG Excellence Initiative of the German Federal and State Governments and CL143/10-1