ModRed conference
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Reduced Basis Modeling for Time-Harmonic Maxwell's Equations

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Computational Methods in Systems and Control Theory

Reduced Basis Method

Model problem:

$$\begin{cases} \text{For } \nu \in \mathcal{D} \subset \mathbb{R}^P, \text{ evaluate} \\ s(\nu) = l(u(\nu);\nu), \\ \text{where } u(\nu) \in X(\Omega) \text{ satisfies} \\ a(u(\nu), v; \nu) = f(v;\nu), \forall v \in X \end{cases}$$

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which is generated by rigorous error estimators

$$\|u(\nu) - u_N(\nu)\|_X \leq \Delta_N(\nu) = \frac{\|r(\cdot; \nu)\|_{X'}}{\beta_{LB}(\nu)}.$$

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- Offline-Online decomposition
  Large pre-processing "Offline" cost, but low "Online" cost for each input-output evaluation.

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$$A(\nu)x(\nu) = b(\nu)$$

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which enables the Offline-Online decomposition.

Project system onto low order space by snapshot matrix V_N as

$$A_N^q = V_N^T A^q V_N, \quad b_N^q = V_N^T b^q$$

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In contrast to Proper Orthogonal Decomposition (POD), the RBM uses rigorous error estimators to generate the reduced order space V_N.

Greedy sampling

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For $N = 2, ..., N_{max}$, find $\nu^N = \arg\max_{\nu \in \Xi} \Delta_{N-1}(\nu)$,

then set $S_N = S_{N-1} \cup \nu^N, \quad V_N = V_{N-1} + span\{u(\nu^N)\}$.

Successive Constraint Method

Output error estimator $\Delta_N^o = \frac{\|r^{pr}(\cdot;\nu)\|_{X'}\|r^{du}(\cdot;\nu)\|_{X'}}{\beta_{LB}(\nu)}$.

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$$\beta(\nu) = \inf_{w\in X}\sup_{v\in X}\frac{|a(w,v;\nu)|}{\|w\|_X\|v\|_X} = \inf_{w\in X}\frac{\|a(w,\cdot;\nu)\|_{X'}}{\|w\|_X}$$

Employing a Successive Constraint Method to determine lower bounds on the stability constant

$$\beta_{LB}(\nu) \leq \beta(\nu),$$

poses a significant computational obstacle.

1. Introduction to Reduced Basis Method
2. Model Problems
   - Simulation of Electric Fields in Semiconductor Models
   - Geometric Variation
3. Simulation Results
4. Taylor Reduced Basis

Constitutive Equations

Full time-harmonic Maxwell's equations

$$\mu^{-1}(\nabla \times E, \nabla \times v) + i\omega\sigma(E, v) - \omega^2\epsilon(E, v) = i\omega J \quad \forall v \in X$$

$$E \times n = 0 \qquad \text{on } \Gamma_{\text{PEC}} \cup \Gamma_{\text{conductor}}$$

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$$a(w, v; \nu) = \sum_{q=1}^{Q} \Theta_q(\nu) a_q(w, v)$$

Matrices in the affine form will also be real symmetric.

Model Problem

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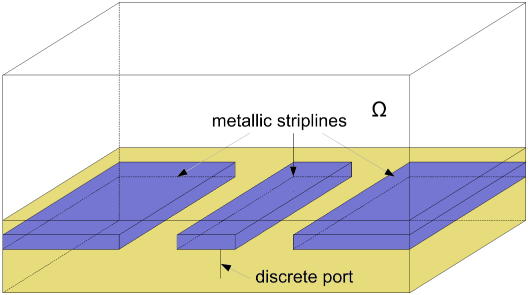

Figure: Geometry of coplanar waveguide.

Geometric Parameters

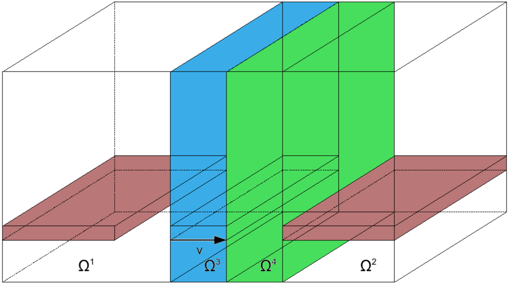

Figure: Affected subdomains by geometric parameter ν.

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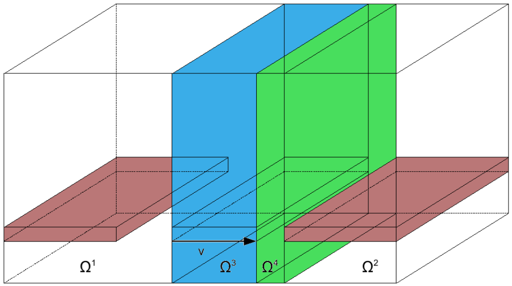

Figure: Affected subdomains by geometric parameter ν.

Given affine mappings from a reference configuration $\Omega^k(\overline{\nu})$ to the actual configuration $\Omega^k(\nu)$

$$T^k : \Omega^k(\overline{\nu}) \to \Omega^k(\nu) : x \mapsto y = G(\nu)x + D(\nu),$$

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the PDE can be solved by a transformation to the reference domain

$$\begin{aligned}\int_{\Omega(\nu)=T(\Omega(\overline{\nu}))} & \mu^{-1}(\nabla \times E, \nabla \times v) + i\omega\sigma(E, v) - \omega^2\epsilon(E, v) \\ = \int_{\Omega(\overline{\nu})} & \mu^{-1} G^{-1}(\nu)(\nabla \times E, \nabla \times v) G^{-T}(\nu) |\det G(\nu)| \\ & + G^T(\nu)\left(i\omega\sigma(E, v) - \omega^2\epsilon(E, v)\right) G(\nu) \frac{1}{|\det G(\nu)|}\end{aligned}$$

i.e. no remeshing is required.

Branchline Coupler

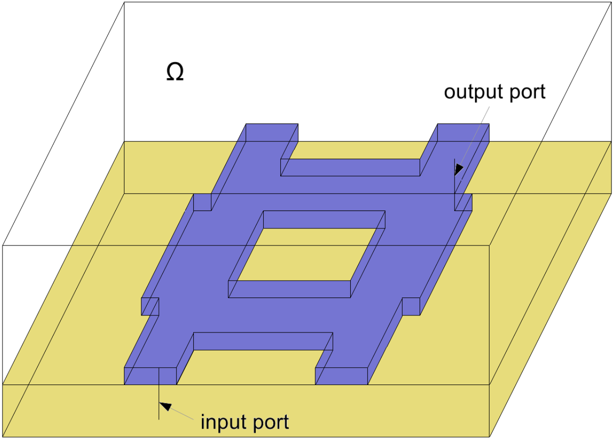

Figure: Geometry of branchline coupler.

Petrov-Galerkin Reduced Basis

Microstrip line models like the branchline coupler contain resonances, i.e. frequencies at which $A(\nu)$ is singular.
It holds: $A(\nu)$ singular $\iff \beta(\nu) = 0$.

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It holds: $A(\nu)$ singular $\iff \beta(\nu) = 0$.

Using the same space to project the trial and test space can result in $\beta_N(\nu) = 0$ while $\beta(\nu) > 0$.

$$T^\nu : X \to X : (T^\nu w, v)_X = a(w, v; \nu) \quad \forall v \in X$$

$$V_N = \text{span}\{u(\nu_1), u(\nu_2), ..., u(\nu_N)\},$$

$$W_N^\nu = \text{span}\{T^\nu u(\nu_1), T^\nu u(\nu_2), ..., T^\nu u(\nu_N)\},$$

$\implies \beta_N(\nu) \geq \beta(\nu)$

Petrov-Galerkin Reduced Basis

Full system

$$A(\nu)x(\nu) = b(\nu)$$
$$y(\nu) = c^T x(\nu)$$

is projected as

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$$A_N(\nu) = {W_N^{\nu}}^T A(\nu) V_N = \sum_{q=1}^{Q_a} \sum_{q'=1}^{Q_a} \Theta_a^q(\nu) \Theta_a^{q'}(\nu) {W_N^{q'}}^T A^q V_N.$$

Simulation Results Coplanar Waveguide

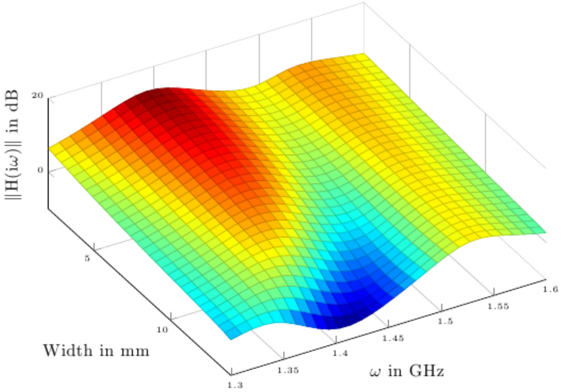

Figure: Transfer function in $[1.3, 1.6]$ GHz $\times[2.0, 14.0]$ mm. Full model contains 52134 dofs. Simulation with full model takes 14740s, reduced model 10s (N=85).

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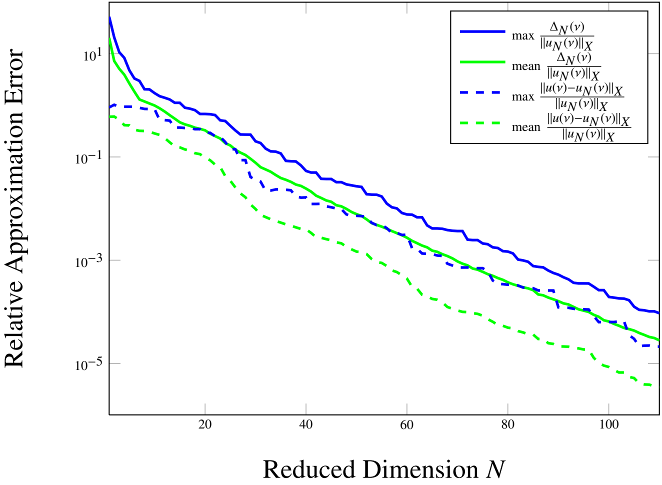


Figure: Convergence History.

Simulation Results Branchline Coupler

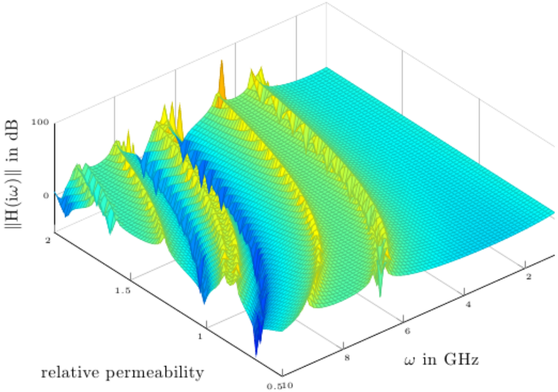
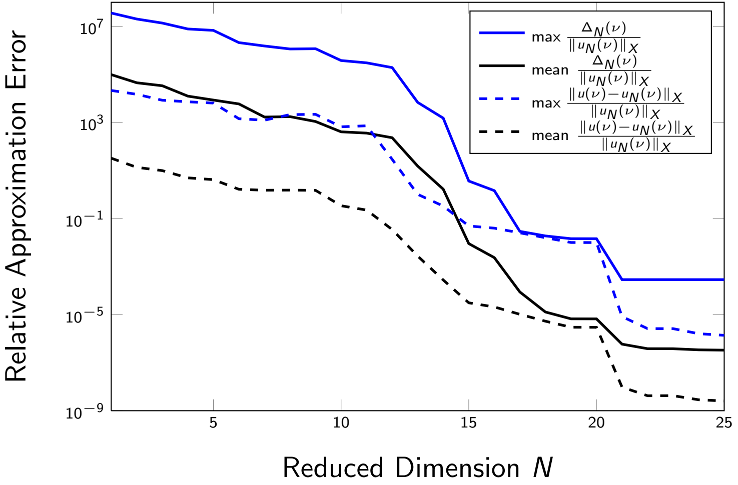


Figure: Transfer function in $[1.0, 10.0]$ GHz $\times[0.5, 2.0]$. Full model contains 27679 dofs. Simulation with full model takes 8644s, reduced model 1s (N=25).

Simulation Results Branchline Coupler



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2 Model Problems

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4 Taylor Reduced Basis
- Numerical Results

Taylor Reduced Basis

The Taylor Reduced Basis space takes the derivatives of snapshot solutions into account. Thus the RB space V_N is defined as

$$V_N = \{u(\nu_1), \partial_\omega u(\nu_1), \partial_p u(\nu_1), \dots, u(\nu_N), \partial_\omega u(\nu_N), \partial_p u(\nu_N)\},$$

where ∂_ω and ∂_p denote partial derivatives with respect to the frequency ω and the geometric parameter p.

The linear system to be solved for a snapshot ν is

$$A(\nu)x(\nu) = b,$$

with parameter-dependent system matrix $A(\nu)$ and right hand side b. This allows to compute the derivative $\partial_\omega x(\nu)$ by

$$\begin{aligned} \partial_\omega(A(\nu)x(\nu)) &= \partial_\omega(b) \\ (\partial_\omega A(\nu))x(\nu) + A(\nu)(\partial_\omega x(\nu)) &= 0 \end{aligned}$$

leading to the linear system

$$A(\nu)(\partial_\omega x(\nu)) = -(\partial_\omega A(\nu))x(\nu).$$

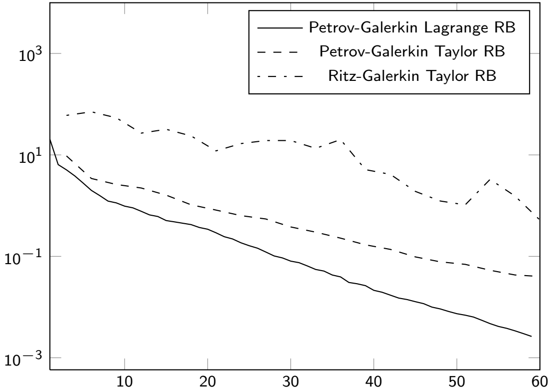

Figure: Mean relative RB approximation error estimator for parametric variation of frequency from 1.3 GHz to 1.6 GHz and middle stripline width from 2mm to 14mm.

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