Solving parametric algebraic Lyapunov equations using reduced basis method

with application to PMOR

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Outline

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PMOR by balanced truncation

$$
\begin{aligned}
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y &= C(p)x
\end{aligned}
$$

PMOR by balanced truncation

$$\begin{array}{rcl} E(p)\,\dot{x} & = & A(p)\,x + B(p)\,u \\ y & = & C(p)x \end{array} \quad \xrightarrow{\text{PMOR}} \quad \begin{array}{rcl} \hat{E}(p)\,\dot{\hat{x}} & = & \hat{A}(p)\,\hat{x} + \hat{B}(b)\,u, \\ \hat{y} & = & \hat{C}(p)\hat{x},\;\; \hat{x} \in \mathbb{R}^r,\; r \ll N. \end{array}$$

- preservation of system properties (stability, passivity, ...)
- preservation of the parameter dependence
- small approximation error

$$\|C(p)(sE(p) - A(p))^{-1}B(p) - \hat{C}(p)(s\hat{E}(p) - \hat{A}(p))^{-1}\hat{B}(p)\|$$
$$=: \|H - \hat{H}\| \leq tol.$$

Problem statement

- Lyapunov equations for balanced truncation

$$A(p)X(p)E^T(p) + E(p)X(p)A^T(p) = -B(p)B^T(p), \text{ control.}$$
$$A^T(p)Y(p)E(p) + E^T(p)Y(p)A(p) = -C^T(p)C(p), \text{ obser.}$$

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- **Problem statement**: find the solution $X(p)$ of

$$A(p)X(p)E^T(p)+E(p)X(p)A^T(p) = -B(p)B^T(p), \; p \in \mathbb{P} \subset \mathbb{R}^d$$

with $E(p)$ symmetric positive definite, $A(p)$ symmetric negative definite for all p in closed, bounded $\mathbb{P}$.

Outline

Lyapunov eq. + Kronecker prod. = lin. system

- Lyapunov equation

$$-A(p)X(p)E^T(p) - E(p)X(p)A^T(p) = B(p)B^T(p),$$

where $E(p), A(p), X(p) \in \mathbb{R}^{N \times N}$, $B(p) \in \mathbb{R}^{N \times m}$, $m \ll N$.

- Linear system:

$$\mathbf{L}(p)\,\mathbf{x}(p) = \mathbf{b}(p),$$

where $\mathbf{L}(p) = -E(p) \otimes A(p) - A(p) \otimes E(p) \in \mathbb{R}^{N^2 \times N^2}$,

$$\mathbf{x}(p) = \mathrm{vec}(X(p)), \mathbf{b}(p) = \mathrm{vec}(B(p)B^T(p)) \in \mathbb{R}^{N^2}$$

RB method for linear systems

[Patera/Rozza'07, also Maday, Grepl, Haasdonk, Ohlberger, Urban, ...]

Reduced basis method for $\mathbf{L}(p)\mathbf{x}(p) = \mathbf{b}(p)$

- Snapshots collection:
  construct the reduced basis matrix
  $\mathbf{V}_k = [\mathbf{x}(p_1), \ldots, \mathbf{x}(p_k)] \in \mathbb{R}^{N^2 \times k}$, $k \ll N^2$
- Galerkin projection:
  approximate the solution $\mathbf{x}(p) \approx \mathbf{V}_k \widetilde{\mathbf{x}}(p)$, where $\widetilde{\mathbf{x}}(p) \in \mathbb{R}^k$
  solves
  $\widetilde{\mathbf{L}}(p)\widetilde{\mathbf{x}}(p) = \widetilde{\mathbf{b}}(p)$ with $\widetilde{\mathbf{L}}(p) = \mathbf{V}_k^T \mathbf{L}(p)\mathbf{V}_k$, $\widetilde{\mathbf{b}}(p) = \mathbf{V}_k^T \mathbf{b}(p)$
- **Questions**
  - How to choose the parameters $p_1, \ldots, p_k$?
  - How to estimate the error $\mathbf{e}_k(p) = \mathbf{x}(p) - \mathbf{V}_k \tilde{\mathbf{x}}(p)$?
  - How to make the computations efficient?

More assumptions

- Affine dependence:

$$E(p) = \sum_{i=1}^{n_E} \theta_i^E(p) E_i, \quad A(p) = \sum_{i=1}^{n_A} \theta_i^A(p) A_i,$$

$$B(p) = \sum_{i=1}^{n_B} \theta_i^B(p) B_i, \quad C(p) = \sum_{i=1}^{n_C} \theta_i^C(p) C_i.$$

- Parametric coercivity:

$$E_i = E_i^T > 0, \; \theta_i^E(p) > 0, \; A_i = A_i^T < 0, \; \theta_i^A(p) > 0$$

$$\hookrightarrow \alpha(p) := \inf_{\mathbf{v} \in \mathbb{R}^{N^2} \setminus \{0\}} \frac{\mathbf{v}^T \mathbf{L}(p) \mathbf{v}}{\|\mathbf{v}\|^2} > 0$$

Error analysis: min-θ approach

Let $\mathbf{e}_k(p) = \mathbf{x}(p) - \mathbf{V}_k\tilde{\mathbf{x}}(p)$, $\mathbf{r}_k(p) = \mathbf{b}(p) - \mathbf{L}\mathbf{V}_k\tilde{\mathbf{x}}(p)$,
$\gamma(p) := \sup\limits_{\mathbf{w},\mathbf{v}\in\mathbb{R}^{N^2}\setminus\{0\}} \dfrac{\mathbf{w}^T\mathbf{L}(p)\mathbf{v}}{\|\mathbf{w}\|\|\mathbf{v}\|} (< \infty)$. With fixed $\bar{p} \in \mathbb{P}$,

$$\theta^{\bar{p}}_{\min}(p) = \min_{\substack{i=1,\cdots,n_A\\ j=1,\cdots,n_E}} \frac{\theta^A_i(p)\theta^E_j(p)}{\theta^A_i(\bar{p})\theta^E_j(\bar{p})},$$

$$\theta^{\bar{p}}_{\max}(p) = \max_{\substack{i=1,\cdots,n_A\\ j=1,\cdots,n_E}} \frac{\theta^A_i(p)\theta^E_j(p)}{\theta^A_i(\bar{p})\theta^E_j(\bar{p})}, \theta^{\bar{p}}(p) = \frac{\theta^{\bar{p}}_{\max}(p)}{\theta^{\bar{p}}_{\min}(p)}.$$

$$\Delta_k(p) = \frac{||\mathbf{r}_k(p)||}{2\theta^{\bar{p}}_{\min}(p)\lambda_{\min}(A(\bar{p}))\lambda_{\min}(E(\bar{p}))} =: \frac{||\mathbf{r}_k(p)||}{\alpha_{LB}(p)}.$$

Error analysis: min-θ approach *ctd.*

Theorem

$$||e_k(p)|| \leq \Delta_k(p) \leq ||e_k(p)||\, \theta^{\bar{p}}(p) cond(A(\bar{p})) cond(E(\bar{p})).$$

Use(s) of error estimate:

- Error control
- Construction of reduced basis via (so-called) Greedy algorithm

Error analysis: min-θ approach *ctd.*

Theorem

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Greedy algorithm

Input: tolerance *tol*, train set $\Xi \subset \mathbb{P}$, initial parameter $p_1 \in \Xi$

Output: Chosen set points $p_1, \ldots, p_k \subset \Xi$, projection matrix V_k.

Greedy algorithm

- Solve $\mathbf{L}(p_1)\mathbf{x}(p_1) = \mathbf{b}(p_1)$
- Set $j = 2$, $\Delta_1^{\max} = 1$ and $\mathbf{V}_1 = \mathbf{x}(p_1)$
- while $\Delta_{j-1}^{\max} \geq tol$

$$p_j = \arg\max_{p \in \Xi} \Delta_{j-1}(p) \qquad \% \;\; \Delta_{j-1}(p) = \frac{\|\mathbf{r}_{j-1}(p)\|}{\alpha_{LB}(p)}$$

$$\Delta_j^{\max} = \Delta_{j-1}(p_j)$$

solve $\mathbf{L}(p_j)\mathbf{x}(p_j) = \mathbf{b}(p_j)$

$$\mathbf{V}_j = [\,\mathbf{V}_{j-1},\; \mathbf{x}(p_j)\,]$$

$$j \leftarrow j + 1$$

end

Computation of the residual

Let $\mathbf{L}(p) = \sum_{i=1}^{n_E n_A} \theta_i^{\mathbf{L}}(p)\,\mathbf{L}_i$ and $\mathbf{b}(p) = \sum_{i=1}^{n_B^2} \theta_i^{\mathbf{b}}(p)\,\mathbf{b}_i$. Then

$$
\begin{aligned}
\|\mathbf{r}_j(p)\|^2 &= \|\mathbf{b}(p) - \mathbf{L}(p)\boldsymbol{V}_j\,\tilde{\mathbf{x}}(p)\|^2 \\
&= \sum_{i,l=1}^{n_B^2} \theta_i^{\mathbf{b}}(p)\,\theta_l^{\mathbf{b}}(p)\,\mathbf{b}_i^T\mathbf{b}_l \\
&\quad -2\sum_{i=1}^{n_B^2}\sum_{l=1}^{n_E n_A} \theta_i^{\mathbf{b}}(p)\,\theta_l^{\mathbf{L}}(p)\,\mathbf{b}_i^T\mathbf{L}_l\mathbf{V}_j\,\tilde{\mathbf{x}}(p) \\
&\quad + \sum_{i,l=1}^{n_E n_A} \theta_i^{\mathbf{L}}(p)\,\theta_l^{\mathbf{L}}(p)\,\tilde{\mathbf{x}}(p)^T\mathbf{V}_j^T\mathbf{L}_i^T\mathbf{L}_l\mathbf{V}_j\,\tilde{\mathbf{x}}(p),
\end{aligned}
$$

where $\widetilde{\mathbf{x}}(p)$ solves $\widetilde{\mathbf{L}}(p)\widetilde{\mathbf{x}}(p) = \widetilde{\mathbf{b}}(p)$ with

$\widetilde{\mathbf{L}}(p) = \sum_{i=1}^{n_E n_A} \theta_i^{\mathbf{L}}(p)\,\mathbf{V}_j^T\mathbf{L}_i\mathbf{V}_j$ and $\widetilde{\mathbf{b}}(p) = \sum_{i=1}^{n_B^2} \theta_i^{\mathbf{b}}(p)\,\mathbf{V}_j^T\mathbf{b}_i$

Offline-online decomposition

- Offline phase: compute all parameter-independent quantities
- Online phase: for given $p \in \Xi$, compute $\tilde{\mathbf{x}}(p)$ and $\|\mathbf{r}_j(p)\|/\alpha_{\mathrm{LB}}(p)$

Notes on computational efficiency:

- No computation with N^2-dim matrices and vectors
- Hierachical structure of parameter-independent quantities
- Low-rank structure of solution $R_j R_j^T$ of Lyapunov equations. E.g.: Computation of $\mathbf{V}_j^T \mathbf{L}_i^T \mathbf{L}_l \mathbf{V}_j$. $\mathbf{v}_h^T \mathbf{L}_i^T \mathbf{L}_l \mathbf{v}_t =$ $\langle -A_k R_h R_h^T E_r^T - E_r R_h R_h^T A_k^T, -A_s R_t R_t^T E_q^T - E_q R_t R_t^T A_s^T \rangle_F$ $= 2\mathrm{trace}\big((A_k R_h)^T (A_s R_t)(E_q R_t)^T (E_r R_h)$ $+ (E_r R_h)^T (A_s R_t)(E_q R_t)^T (A_k R_h)\big)$

Online solution of Lyapunov equation

- Linear combination
  $X(p) \approx \mathrm{vec}^{-1}(\mathbf{V}_j \widetilde{\mathbf{x}}(p)) = \sum_{i=1}^{j} \widetilde{x}_i(p) R_i R_i^T =: X_{\mathrm{RB}}^{I}(p)$ where
  $\widetilde{\mathbf{x}}(p) = [\widetilde{x}_1(p), \ldots, \widetilde{x}_j(p)]^T$ solves $\widetilde{\mathbf{L}}(p)\widetilde{\mathbf{x}}(p) = \widetilde{\mathbf{b}}(p)$.
  $\hookrightarrow X(p)$ may not be positive (semi) definite.
- Projection (with column compression)
  $X(p) \approx U \widetilde{X}(p) U^T$, where

$$\widetilde{A}(p)\widetilde{X}(p)\widetilde{E}^T(p) + \widetilde{E}(p)\widetilde{X}(p)\widetilde{A}^T(p) = -\widetilde{B}(p)\widetilde{B}^T(p) \qquad (1)$$

with $\widetilde{E}(p) = \sum_{i=1}^{n_E} \theta_i^E(p) U^T E_i U$, $\widetilde{A}(p) = \sum_{i=1}^{n_A} \theta_i^A(p) U^T A_i U$,
$\widetilde{B}(p) = \sum_{i=1}^{n_B} \theta_i^B(p) U^T B_i$ and $U = [\, R_1, \ldots, R_j \,]$ (compressed.)

Outline

PMOR by balanced truncation [Tombs '87]

Suppose from (1),

$$
\begin{aligned}
X(p) &\approx U_{\mathcal{R}} \widetilde{X}(p) U_{\mathcal{R}}^T = U_{\mathcal{R}} G_{\mathcal{R}}(p) G_{\mathcal{R}}^T(p) U_{\mathcal{R}}^T \\
Y(p) &\approx U_{\mathcal{O}} \widetilde{Y}(p) U_{\mathcal{O}}^T = U_{\mathcal{O}} G_{\mathcal{O}}(p) G_{\mathcal{O}}^T(p) U_{\mathcal{O}}^T
\end{aligned}
$$

are the approximate controllability, observability gramians. First, compute the SVD

$$
G_{\mathcal{O}}(p)^T U_{\mathcal{O}}^T E(p) U_{\mathcal{R}} G_{\mathcal{R}}(p) = \sum_{q=1}^{n_E} \theta_q^E(p) G_{\mathcal{O}}(p)^T U_{\mathcal{O}}^T E_q U_{\mathcal{R}} G_{\mathcal{R}}(p)
$$

$$
= \begin{bmatrix} U_1(p) & U_2(p) \end{bmatrix} \begin{bmatrix} \Sigma_1(p) & 0 \\ 0 & \Sigma_2(p) \end{bmatrix} \begin{bmatrix} V_1(p) & V_2(p) \end{bmatrix}^T .
$$

PMOR by balanced truncation *ctd.*

Then formulate the reduced order model:

$$\hat{E}(p) = \sum_{q=1}^{n_E} \theta_q^E(p)\Sigma_1(p)^{-1/2}U_1(p)^T G_{\mathcal{O}}(p)^T U_{\mathcal{O}}^T E_q U_{\mathcal{R}} G_{\mathcal{R}}(p)V_1(p)\Sigma_1(p)^{-1/2},$$

$$\hat{A}(p) = \sum_{q=1}^{n_A} \theta_q^A(p)\Sigma_1(p)^{-1/2}U_1(p)^T G_{\mathcal{O}}(p)^T U_{\mathcal{O}}^T A_q U_{\mathcal{R}} G_{\mathcal{R}}(p)V_1(p)\Sigma_1(p)^{-1/2},$$

$$\hat{B}(p) = \sum_{q=1}^{n_B} \theta_q^B(p)\Sigma_1(p)^{-1/2}U_1(p)^T G_{\mathcal{O}}(p)^T U_{\mathcal{O}}^T B_q$$

$$\hat{C}(p) = \sum_{q=1}^{n_C} \theta_q^C(p) C_q U_{\mathcal{R}} G_{\mathcal{R}}(p)V_1(p)\Sigma_1(p)^{-1/2}.$$

Offline-Online decomposition RBPMOR

Offline Given $U_{\mathcal{R}}, U_{\mathcal{O}}$, compute and store all parameter-independent terms: $U_{\mathcal{O}(\mathcal{R})}^T E_q U_{\mathcal{O}(\mathcal{R})}$, $U_{\mathcal{O}(\mathcal{R})}^T A_q U_{\mathcal{O}(\mathcal{R})}$, $U_{\mathcal{R}}^T B_q$, and $C_q U_{\mathcal{O}}$ for reduced Lyapunov equations; $U_{\mathcal{O}}^T E_q U_{\mathcal{R}}$, $U_{\mathcal{O}}^T A_q U_{\mathcal{R}}$, $U_{\mathcal{O}}^T B_q$, and $C_q U_{\mathcal{R}}$ for PMOR.

Online Given $p \in \mathbb{P}$,

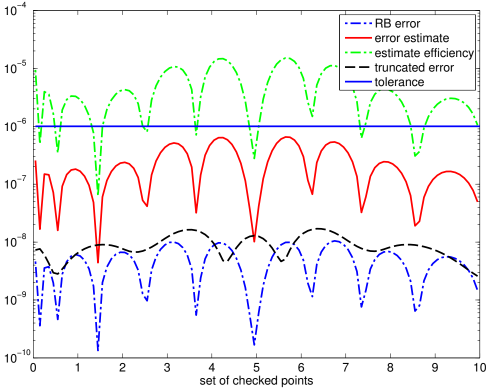
- Compute $\widetilde{A}(p), \widetilde{E}(p), \widetilde{B}(p), \widetilde{C}(p)$
- Solve reduced Lyapunov equations for $G_{\mathcal{R}}(p)$ and for $G_{\mathcal{O}}(p)$
- Compute the SVD and truncate for $\Sigma_1(p), U_1(p)$ and $V_1(p)$
- Compute the reduced order model at p

Outline

Example 1

$E(p) = I_{200}, B \in \mathbb{R}^{200}, \Xi = [0:0.1:10] \times [0:0.1:10]$ and

$$A(p_1, p_2) = (((p_1 - 5)^2 + 2(p_2 - 5)^2)/20 + 1))A_1 + \sin(0.1 + \sqrt{p_1/2})A_2 \in \mathbb{R}^{200\times 200}.$$

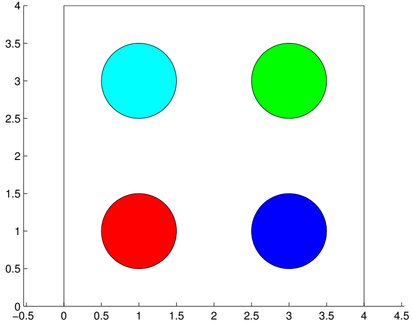


Error, error estimate, $tol_{Greedy} = tol_{SVD} = 1e-6$, Greedy iteration number = 9, compressed 279→40

Example 2: [Kressner/Tobler '11]

$$\begin{aligned} \dot{z} - \nabla(\sigma(\xi)\nabla z) &= f \text{ in } \Omega \\ z &= 0 \text{ on } \partial\Omega \end{aligned} \quad \text{with } \sigma(\xi) = \begin{cases} 1 + p_i, & \xi \in C_i \\ 1, & \xi \in \Omega \setminus (\cup C_i). \end{cases}$$

$E, A(p) \in \mathbb{R}^{1580\times1580}, B, C^T \in \mathbb{R}^{1580}, \Xi = [0.1 : 0.25 : 10.1]^4,$



Example 2: [Kressner/Tobler '11] *ctd.*

Greedy iterations: 20, final reduced order: 50, error checked on 50 randomly chosen points.

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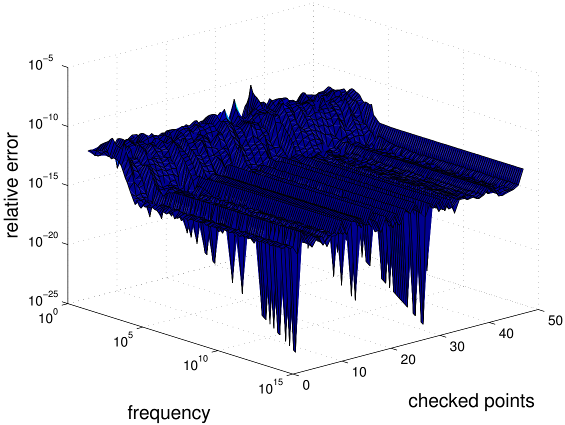

Outline

Conclusion

- Reduced basis method for parametric Lyapunov equations with general $E(p)$ and $A(p)$

  $\hookrightarrow$ computable upper bounds on or $\|\boldsymbol{L}(p)^{-1}\|$ are required
- Other error estimation techniques (succesive constraints method [Huynh/Rozza/Sen/Patera'07], ...)
- Reduced basis method for parametric projected Lyapunov equations

$$E(p)X(p)A^T(p) + A(p)X(p)E^T(p) = -P_l(p)BB^TP_l^T(p)$$
$$X(p) = P_r(p)X(p)P_r^T(p)$$

- Reduced basis method for parametric (projected) Riccati and Lur'e equations