Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods for Systems and Control Theory
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Summer School on Numerical Linear Algebra for Dynamical and High-Dimensional Problems

Model Reduction for Linear Dynamical Systems

Exercise 1 **(Balanced Truncation and Iterative Rational Krylov Algorithm)**

Implement the

1. iterative rational Krylov algorithm,
2. square root balanced truncation method

for a standard state space system (A, B, C). The Lyapunov equations can be solved via the sign function solver discussed in the lecture.

Exercise 2 **(Tunable Optical Filter)**

The first to-be-reduced system is a simplified thermal model of a filter device resulting from a spatial discretization by finite elements. Download the corresponding system matrices (E, A, B, C) on the homepage of the Oberwolfach Model Reduction Benchmark Collection. Since the matrices are given in MatrixMarket format, you will need the routine 'mmread.m' in order to work with the data in Matlab or Octave. Compute the 12 largest Hankel singular values of the system. Subsequently, compute reduced order models of dimension $r = 3$ and $r = 5$ by means of the two algorithms implemented in Exercise 1 (premultiply with E^{-1}). Compare the accuracy by creating Bode plots for full and reduced order systems, i.e. evaluate $||H(i\omega)||_2$, $\omega \in [10^{-4}, 10^4]$. Make sure that you state your results in dB.

Exercise 3 **(Optimal Cooling of Steel Profiles)**

The second example arises in the context of optimal cooling of steel profiles. Once more, the model data ($n = 1357$) is part of the Oberwolfach Model Reduction Benchmark Collection. Why should one avoid the premultiplication with E^{-1} in this case? Modify the sign function solver appropriately such that you can work with the generalized Lyapunov equations. Compute the first 50 Hankel singular values of the system. What has to be taken into account? Modify the algorithms from Exercise 1 such that they can handle generalized state space systems of the form (E, A, B, C). Further, test your routines by computing reduced order models of dimension $r = 5$ and $r = 20$. Use the frequency range $[10^{-2}, 10^6]$ in your Bode plot.

Exercise 4 **(CD Player)**

As a final example we consider the finite element model of a CD player. Download the data and compute reduced order models (balanced truncation and IRKA) varying from system dimensions $r = 2, 4, \ldots, 40$. Evaluate the accuracy in terms of the relative $\mathcal{H}_2$-norm of the error systems. Compare your results within the group. What do you observe and what is the explanation for this?

Exercise 5 (Krylov-Plus-Inverted-Krylov (KPIK))

As a fast and reliable alternative for computing low-rank approximations to the the solution of Lyapunov equations of the form

$$AP + PA^T + BB^T = 0,$$

with $B \in \mathbb{R}^n$ one can use the so-called Krylov-Plus-Inverted-Krylov (KPIK) method. The main idea is to solve a reduced Lyapunov equation of dimension $r \ll n$ which is obtained by an orthogonal projection $\mathcal{P} = VV^T$ onto the column space of V. The approximate solution then is given by prolongation $\hat{P} = VP_rV^T \approx P$. For standard state space systems, the method relies on constructing V as an orthogonal basis of the union of the Krylov spaces $\mathcal{K}_q(A, B)$ and $\mathcal{K}_q(A^{-1}, A^{-1}B)$. Implement an iterative method applicable to generalized state space systems which in each step checks the accuracy in terms of the relative residual $||A\hat{P} + \hat{P}A^T + BB^T||_F/||BB^T||_F$. Try your method for the previous examples, but make sure that you only use right hand sides BB^T with $B \in \mathbb{R}^n$.