Otto-von-Guericke-Universität Magdeburg
Fakultät für Mathematik – Institut für Analysis und Numerik
Max-Planck-Institut-Magdeburg
Computational Methods for Systems and Control Theory

Summer Term 2012
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Model Reduction for Dynamical Systems – 1. Exercise

Deadline for homework: 04/24/2012 in the exercise

Exercise 1 (Controllability of dynamical systems)

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. Let $K = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ be the associated Kalman controllability matrix and let $r := \operatorname{rank}(K)$.

a) Show that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \quad \text{and} \quad \tilde{B} := T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

where (A_1, B_1) is a controllable matrix pair.

b) Show that (A, B) is controllable iff $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ has full row rank $\forall \lambda \in \mathbb{C}$.

Exercise 2 (Lanczos recurrence for Krylov subspaces)

Consider the Lanczos recurrence

$$\beta_k v_{k+1} = Av_k - \alpha_k - \beta_{k-1}v_{k-1},$$

with v_1 arbitrary but $||v_1|| = 1$, $v_0 = 0$, $\alpha_k = (Av_k, v_k)$ and β_k such that $||v_k|| = 1$. prove that this procedure generates an orthonormal basis for the Krylov subspace $\mathcal{K}_k(A, v_1)$.

Hint: Use induction and also that for a symmetric matrix A it holds

$$(Av_k, v_{k-1}) = (v_k, Av_{k-1})$$

as well as that $Av_{k-1} = \beta_{k-1}v_k + w$ with $w \in \operatorname{span}\{v_1, \ldots, v_{k-1}\}$.

Exercise 3 (Homework) (Properties of a Dynamical System)

Consider the dynamical system given by

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Analyze the above system w.r.t. stability, controllability and observability. Further, compute the transfer function $H(s) = c^T(sI - A)^{-1}b$ of the system. What can you say with regard to minimality of the system?

Exercise 4 (Homework) (Properties of the Singular Value Decomposition (SVD))

Let $r = \operatorname{rank}(A)$, $A = U\Sigma V^T$ be the singular value decomposition with $U = \begin{bmatrix} u_1, \ldots, u_m \end{bmatrix} \in \mathbb{R}^{m\times m}$, $V = \begin{bmatrix} v_1, \ldots, v_n \end{bmatrix} \in \mathbb{R}^{n\times n}$ orthonormal. Show the following statements:

a) *Schmidt-Eckart-Young-Mirsky-Theorem*: If $k < r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, then it holds:

$$\min_{\text{rank}(B)=k} ||A - B||_2 = ||A - A_k||_2 = \sigma_{k+1},$$

that means A_k is the best rank-k approximation of A.

Hint: Realize that $\text{rank}\,(A_k) = k$. Then select a matrix $B \in \mathbb{R}^{m \times n}$ with $\text{rank}\,(B) = k$. It exist orthonormal vectors $x_1, \ldots, x_{n-k}$ such that $\ker(B) = \text{span}\{x_1, \ldots, x_{n-k}\}$ and it holds: $\text{span}\{x_1, \ldots, x_{n-k}\} \cap \text{span}\{v_1, \ldots, v_{k+1}\} \neq \{\emptyset\}$. Let z be a vector of this intersection with $||z|| = 1$. Then it holds $Bz = 0$. Making use of the representation of Az one can show the necessary estimate.

b) It holds:

$$\begin{aligned} \ker(A) =& \{v_{r+1}, \ldots, v_n\}, \\ \text{range}\,(A) =& \quad \{u_1, \ldots, u_r\}. \end{aligned}$$

c) It holds:

$$||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}.$$

Exercise 5 (Homework) (Image Data Compression via SVD)

Write a MATLAB routine which reads in a matrix containing image data, computes a best rank-k approximation of this matrix and then displays the approximation error (w.r.t. the 2-norm), original and compressed image as well as required memory for storing the images. Try your program by means of a picture of the Cathedral of Magdeburg which you can find as *dom_md_c.mat* on the course homepage. Test different values of k and empirically determine the smallest value of k which can be used without a visible loss of accuracy.

Hint: The MATLAB image is created with the *load* command, e.g.,

```
>> load dom_md_c.mat
```

yields an image data matrix of dimension 200×300. For a visualization of the image in MATLAB, you can use the commands

```
>> colormap(map)
>> image(A).
```

Send your routines to breiten@mpi-magdeburg.mpg.de. **The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name_ha1a5. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.**