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Fakultät für Mathematik – Institut für Analysis und Numerik
Max-Planck-Institut für Dynamik komplexer technischer Systeme
Computational Methods for Systems and Control Theory

Summer Term 2012
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Model Reduction for Dynamical Systems – 2. Exercise

Deadline for homework: 05/15/2012 in the exercise

Exercise 1 **(Balanced realizations)**

Given a minimal LTI system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x^0, \\ y(t) &= Cx(t) + Du(t). \end{aligned}$$

Show that a balanced realization is given by the state-space transformation

$$T_b := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$ and $Q = R^T R$ (e.g., Cholesky decompositions) satisfy the pair of Lyapunov equations

$$\begin{aligned} AP + PA^T + BB^T &= 0, \\ A^T Q + QA + C^T C &= 0 \end{aligned}$$

and

$$SR^T = U \Sigma V^T$$

is the SVD of SR^T.

Hint: First note that $T^{-1} = S^T U \Sigma^{-\frac{1}{2}}$, then the result follows by simple algebraic manipulations.

Exercise 2 **(Computation of system norms)**

a) Consider the system from Exercise 3, exercise sheet 1:

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; d = 0.$$

Analytically compute the $\mathcal{H}_\infty$-norm of the system.

b) Consider the following LTI system:

$$A = \begin{bmatrix} -8 & 8 \\ -8 & -42 \end{bmatrix}, \; b = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \; c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \; d = 0.$$

Analytically compute the $\mathcal{H}_2$-norm of the system.

Hint: Make use of the eigenvalue decomposition of $A = Q \Lambda Q^{-1}$ and the fact that the $\mathcal{H}_2$-norm is invariant under state-space transformations. Further use that the $\mathcal{H}_2$-norm is given as $\sqrt{c^T P c}$, where P satisfies the Lyapunov equation from Exercise 1.

Exercise 3 (Homework) **(Minimality of LTI systems)**

Consider an LTI system as in Exercise 1. Show that the realization (A, B, C, D) is minimal if and only if (A, B) is controllable and (A, C) is observable.

Hint: W.l.o.g. rewrite the system by means of the Kalman controllability decomposition shown in Exercise sheet 1. For observability of (A, C) make use of the Kalman observability decomposition

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_1 & 0 \\ A_2 & A_3 \end{bmatrix} \quad \text{and} \quad \tilde{C} := CT = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

which yields an observable matrix pair (A_1, C_1).

Exercise 4 (Homework) **(The Lyapunov equation)**

Consider the *(infinite) controllability Gramian* $P := \int_0^\infty e^{As}BB^Te^{A^Ts}\,ds$. Assume that (A, B) is controllable. Show that the following two statements are equivalent:

a) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is asymptotically stable.

b) It holds $P > 0$ and $AP + PA^T + BB^T = 0$.

Hint b) $\Rightarrow$ a): Consider an eigenvalue λ of A together with its corresponding eigenvector x. Then pre- and postmultiply the above equation by x^* and x, respectively.

Exercise 5 (Homework) **(Model order reduction by the Arnoldi iteration)**

Write a MATLAB routine which reads in an LTI realization (A, B, C, D), computes an orthogonal basis $V = [v_1, \ldots, v_k]$ for the Krylov subspace $\mathcal{K}_k(A, B)$ by means of the Arnoldi iteration discussed in the lecture and constructs a reduced-order realization as

$$\hat{A} = V^TAV,\ \hat{B} = V^TB,\ \hat{C} = CV,\ \hat{D} = D.$$

Try your program by means of the model of a CD player which you find as *CDPlayer.mat* on the course homepage. Evaluate the transfer function

$$H(i\omega) = C(i\omega I - A)^{-1}B + D$$

for original and reduced-order model over the frequency interval $\omega \in [10^{-1}, 10^5]$. Use 10 000 logarithmically distributed sample points. Plot the *gain* of the transfer function, i.e. $20 \cdot \log_{10} |(H(j\omega)|$ on a logarithmic x-scale by using the MATLAB command *semilogx(ω, H_ω)*. Test several values of k and interpret the results.

Send your routines to breiten@mpi-magdeburg.mpg.de. **The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name ha2a5. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.**