Otto-von-Guericke-Universität Magdeburg
Fakultät für Mathematik – Institut für Analysis und Numerik
Max-Planck-Institut für Dynamik komplexer technischer Systeme
Computational Methods for Systems and Control Theory

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Model Reduction for Dynamical Systems – 6. Exercise

Deadline for homework: 26/05/2012 in the exercise

Exercise 1 (Homework) (Properties of the matrix sign function)

Assume $Z \in \mathbb{C}^{n\times n}$ with no eigenvalues on the imaginary axis. Show that it holds:

a) $\text{sign}(Z)^2 = I_n$, i.e., $\text{sign}(Z)$ is a square root of the identity matrix;

b) $\text{sign}(T^{-1}ZT) = T^{-1}\text{sign}(Z)T$ for all nonsingular $T \in \mathbb{C}^{n\times n}$;

c) if Z is stable, then

$$\text{sign}(Z) = -I_n, \quad \text{sign}(-Z) = I_n.$$

d) Define $Z_0 \leftarrow Z,\ Z_{k+1} \leftarrow \frac{1}{2}(Z_k + Z_k^{-1}), \quad k = 0, 1, 2, \ldots$

Show that the above scheme convergences to $\text{sign}(Z)$.

Exercise 2 (Homework) (Solving Sylvester equations via the matrix sign function)

a) Consider the Sylvester equation

$$AX + XB + C = 0, \tag{1}$$

with $A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{m\times m}$ and $C \in \mathbb{R}^{n\times m}$. Assume that A and B are asymptotically stable matrices and that X is a solution of eq. (1). Show that it holds

$$\text{sign}\left(\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}\right) = \begin{bmatrix} -I & 2X \\ 0 & I \end{bmatrix}.$$

b) Show that instead of iterating on $\begin{bmatrix} A & C \\ 0 & -B \end{bmatrix}$, one can compute X via an iteration on A, B, C.

Exercise 3 (Homework) (Solving algebraic Riccati equations via the matrix sign function)

Motivated by balancing-related methods such as LQG balanced truncation, let us consider the algebraic Riccati equation

$$A^T X + XA - XFX + G = 0,$$

where $A \in \mathbb{R}^{n\times n}$ and $F = F^T,\ G = G^T \in \mathbb{R}^{n\times n}$ are symmetric positive semi-definite matrices and (A, F) is stabilizable. Let $M = \begin{bmatrix} A^T & G \\ F & -A \end{bmatrix}$ and assume that the matrix sign function of M is partitioned as $\text{sign}(M) = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$. Show that it holds $\begin{bmatrix} Z_{11} - I_n \\ Z_{21} \end{bmatrix} X = -\begin{bmatrix} Z_{12} \\ Z_{22} - I_n \end{bmatrix}$.

Hint: First show that it holds

$$M = \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix} \begin{bmatrix} (A - FX)^T & 0 \\ 0 & -(A - FX) \end{bmatrix} \begin{bmatrix} I_n - XQ & X \\ -Q & I_n \end{bmatrix}^{-1},$$

where Q solves $(A - FX)Q + Q(A - FX)^T + F = 0$. Then make use of the properties of the matrix sign function.