Otto-von-Guericke-Universität Magdeburg | Summer Term 2012
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Computational Methods for Systems and Control Theory

Model Reduction for Dynamical Systems – 7. Exercise

Deadline for homework: 07/03/2012 in the exercise

Exercise 1 (Homework) **(The rational Krylov method for the Lyapunov equation)**

Consider the Lyapunov equation

$$AX + XA^T + bb^T = 0,$$

with $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$. Assume that a sequence of $m+1$ shift parameters $S = \{s_1, \ldots, s_m, s_{m+1}\} \subset \mathbb{C}$ is given and that a projection matrix $V \in \mathbb{R}^{n\times m}$ is constructed according to the following procedure.

Algorithmus 1 Rational Krylov Method by Ruhe

Input: A, b, S
Output: $V_{m+1} \in \mathbb{R}^{n\times m+1},\ T_m = V_m^* A V_m \in \mathbb{R}^{m\times m},\ H \in \mathbb{R}^{m+1\times m}$.

Compute $\tilde{v}_1 = (A - s_1 I)^{-1} b$.
Set $v_1 = \tilde{v}_1 / ||\tilde{v}_1||$.
for $j = 2 : m+1$ **do**
  $r = (A - s_j I)^{-1}\ v_{j-1}$.
  $h_{j-1} = V_{j-1}^* r$.
  $r = r - V_{j-1} h_{j-1}$
  $h_{j,j-1} = ||r||$.
  $v_j = r / h_{j,j-1}$.
end for
Set $T_m = V_m^* A V_m$.

Let $D_m = \mathrm{diag}(s_2, \ldots, s_{m+1}) \in \mathbb{R}^{m\times m}$. Show that it holds

$$T_m = (I_m + H_m D_m - V_m^* A v_{m+1} h_{m+1,m} e_m^T) H_m^{-1}.$$

Here, e_m denotes the m-th unit vector and H_m consists of the first m columns of the matrix H.

Assume now that Y_m is a solution of the reduced Lyapunov equation

$$T_m Y_m + Y_m T_m^T + V_m^* b b^T V_m = 0.$$

Show that if $\tilde{v}_1 = b$, the residual $R_m = A V_m Y_m V_m^* + V_m Y_m V_m^* A^T + bb^T$ satisfies

$$||R_m||_F = ||SJS^T||_F,\ J = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

where S is the 3×3 upper triangular matrix in the QR factorization of

$$U = \begin{bmatrix} v_{m+1} s_{m+1} & V_m Y_m H_m^{-T} e_m h_{m+1,m} & -(I - V_m V_m^*) A v_{m+1} \end{bmatrix}.$$

Exercise 2 (Homework) **(The extended Krylov subspace method for the Lyapunov equation)**

Although the above method is easily implementable, the approximation to the solution of the Lyapunov equation heavily depends on the non-trivial choice of the shift parameters. As a fast and reliable alternative for computing low-rank approximations one can use the so-called Krylov-Plus-Inverted-Krylov (KPIK) method. The main idea is to construct V as an orthogonal basis of the union of the Krylov spaces $\mathcal{K}_q(A,b)$ and $\mathcal{K}_q(A^{-1},A^{-1}b)$. Implement an iterative method applicable to (A,b) which in each step checks the accuracy in terms of the relative residual $||AY_{2q}+Y_{2q}A^T+bb^T||_F/||bb^T||_F$. Try your method for the CD player model for different values of q. Compare your results with the method from Exercise 1 with shifts parameters uniformly chosen in the interval $[0,1000]$.

Send your routine to breiten@mpi-magdeburg.mpg.de. **The filename should include your name and the corresponding exercise sheet number as well as the exercise number, e.g., name ha7a2. In case of several files please hand in a compressed file. Moreover, please print the source code of your routine and hand it in together with the other exercises.**