Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lecture 11 —

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de

www.mpi-magdeburg.mpg.de/research/groups/csc/lehre/2012_SS_MOR/

Outline

Padé Approximation

Idea:

- Consider

$$\dot{x} = Ax + Bu, \quad y = Cx$$

with transfer function $G(s) = C(sI_n - A)^{-1}B$.

- For $s_0 \notin \Lambda(A)$:

$$\begin{aligned} G(s) &= C\left(I + (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\ &= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots \end{aligned}$$

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- For $s_0 \notin \Lambda(A)$:

$$\begin{aligned} G(s) &= C((s_0 I_n - A) + (s - s_0)I_n)^{-1}B \\ &= C\left(I + (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\ &= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \dots \end{aligned}$$

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  - For $s_0 = 0$: $m_j := -C(A^{-1})^{j+1}B =$ moments.
  - For $s_0 = \infty$: $m_j := CA^{j-1}B =$ Markov parameters.

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$$\begin{aligned} G(s) &= C\left(I + (s - s_0)(s_0 I_n - A)^{-1}\right)^{-1}(s_0 I_n - A)^{-1}B \\ &= m_0 + m_1(s - s_0) + m_2(s - s_0)^2 + \ldots \end{aligned}$$
- As reduced-order model use rth Padé approximant $\hat{G}$ to G:
$$G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^{2r}),$$
i.e., $m_j = \widehat{m}_j$ for $j = 0, \ldots, 2r - 1$

$\rightsquigarrow$ moment matching if $s_0 < \infty$,

$\rightsquigarrow$ partial realization if $s_0 = \infty$.

Padé Approximation

Asymptotic Waveform Evaluation (AWE):

With

$$\hat{G}(s) = \frac{\alpha_{r-1}s^{r-1} + \alpha_{r-2}s^{r-2} + \ldots + \alpha_1 s + \alpha_0}{\beta_r s^r + \beta_{r-1}s^{r-1} + \ldots + \beta_1 s + 1},$$

the solution of the Padé approximation problem is obtained via solving

$$M \begin{bmatrix} \beta_r \\ \vdots \\ \beta_1 \end{bmatrix} = \begin{bmatrix} m_r \\ \vdots \\ m_{2r-1} \end{bmatrix},$$

with the Hankel matrix $M = \begin{bmatrix} m_0 & m_1 & m_2 & \cdots & m_{r-1} \\ m_1 & m_2 & & ⋰ & m_r \\ m_2 & & ⋰ & ⋰ & \vdots \\ \vdots & ⋰ & ⋰ & ⋰ & \\ m_{r-1} & m_r & \ldots & & m_{2r-2} \end{bmatrix}$.

Then, with $\beta_0 := 1$: $\alpha_j = \sum_{k=0}^{j} m_k \beta_{j-k}$.

Padé Approximation

Padé-via-Lanczos Method (PVL)

- Moments need not be computed explicitly; moment matching is equivalent to projecting state-space onto

$$\mathcal{V} = \mathrm{span}(\tilde{B}, \tilde{A}\tilde{B}, \ldots, \tilde{A}^{r-1}\tilde{B}) =: \mathcal{K}(\tilde{A}, \tilde{B}, r)$$

(where $\tilde{A} = (s_0 I_n - A)^{-1}$, $\tilde{B} = (s_0 I_n - A)^{-1} B$) along

$$\mathcal{W} = \mathrm{span}(C^T, \tilde{A}^* C^T, \ldots, (\tilde{A}^*)^{r-1} C^T) =: \mathcal{K}(\tilde{A}^*, C^T, r).$$

- Computation via unsymmetric Lanczos method, yields system matrices of reduced-order model as by-product.

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Remark: Arnoldi (PRIMA) yields only $G(s) = \hat{G}(s) + \mathcal{O}((s - s_0)^r)$.

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Difficulties:

- Computable error estimates/bounds for $\|y - \hat{y}\|_2$ often very pessimistic or expensive to evaluate.
- Mostly heuristic criteria for choice of expansion points.
  Optimal choice for second-order systems with proportional/Rayleigh damping (Beattie/Gugercin '05).
- Good approximation quality only locally.
- Preservation of physical properties only in special cases; usually requires post processing which (partially) destroys moment matching properties.

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Interpolatory Model Reduction

Short Introduction

Computation of reduced-order model by projection

Given an LTI system $\dot{x} = Ax + Bu,\ y = Cx$ with transfer function $G(s) = C(sI_n - A)^{-1}B,$ a reduced-order model is obtained using projection approach with $V, W \in \mathbb{R}^{n \times r}$ and $W^T V = I_r$ by computing

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV.$$

Petrov-Galerkin-type (two-sided) projection: $W \neq V$,

Galerkin-type (one-sided) projection: $W = V$.

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Rational Interpolation/Moment-Matching

Choose V, W such that

$$G(s_j) = \hat{G}(s_j), \quad j = 1, \ldots, k,$$

and

$$\frac{d^i}{ds^i} G(s_j) = \frac{d^i}{ds^i} \hat{G}(s_j), \quad i = 1, \ldots, K_j, \quad j = 1, \ldots, k.$$

Interpolatory Model Reduction

Short Introduction

Theorem (simplified) [Grimme '97, Villemagne/Skelton '87]

If

$$\begin{aligned} \mathrm{span}\left\{(s_1 I_n - A)^{-1}B, \ldots, (s_k I_n - A)^{-1}B\right\} &\subset \mathrm{Ran}(V), \\ \mathrm{span}\left\{(s_1 I_n - A)^{-T}C^T, \ldots, (s_k I_n - A)^{-T}C^T\right\} &\subset \mathrm{Ran}(W), \end{aligned}$$

then

$$G(s_j) = \hat{G}(s_j), \quad \frac{d}{ds}G(s_j) = \frac{d}{ds}\hat{G}(s_j), \quad \text{for } j = 1, \ldots, k.$$

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Remarks:

using Galerkin/one-sided projection yields $G(s_j) = \hat{G}(s_j)$, but in general

$$\frac{d}{ds}G(s_j) \neq \frac{d}{ds}\hat{G}(s_j).$$

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Remarks:

$k = 1$, standard Krylov subspace(s) of dimension $K \rightsquigarrow$ moment-matching methods/Padé approximation,

$$\frac{d^i}{ds^i}G(s_1) = \frac{d^i}{ds^i}\hat{G}(s_1), \quad i = 0, \ldots, K-1(+K).$$

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Remarks:

computation of V, W from rational Krylov subspaces, e.g.,

- dual rational Arnoldi/Lanczos [Grimme '97],
- Iterative Rational Krylov-Algo. [Antoulas/Beattie/Gugercin '07].