Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lecture 12 —

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Outline

Interpolatory Model Reduction

Padé Approximation

Theorem (moment matching) [Grimme '97, Villemagne/Skelton '87]

For LTI system given by $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $s_1, \ldots, s_k \notin \Lambda(A)$, let

$$A^{(i)} := (s_i I_n - A)^{-1}, \quad B^{(i)} := (s_i I_n - A)^{-1} B, \quad C^{(i)} := C(s_i I_n - A)^{-1}, \ i = 1, \ldots, k,$$

and

$$\begin{aligned} \mathcal{K}_{K_1}\left(A^{(1)}, B^{(1)}\right) \cup \cdots \cup \mathcal{K}_{K_k}\left(A^{(k)}, B^{(k)}\right) &\subset \operatorname{range}(V), \\ \mathcal{K}_{K_1}\left\{(A^{(1)})^T, (C^{(1)})^T\right) \cup \cdots \cup \mathcal{K}_{K_k}\left((A^{(k)})^T, (C^{(k)})^T\right) &\subset \operatorname{range}(W). \end{aligned}$$

Then

$$\frac{d^j}{ds^j} G(s_i) = \frac{d^j}{ds^j} \hat{G}(s_i), \quad \text{for } i = 1, \ldots, k, \ j = 0, \ldots \ell_i - 1,$$

where $\ell_i \geq \lfloor \frac{K_i}{m} \rfloor + \lfloor \frac{K_i}{p} \rfloor$.

Note: for $m \neq p$, must extend $\mathcal{V}$ (for $m < p$) or $\mathcal{W}$ (for $m > p$) by arbitrary vectors yielding an orthogonal bases of the respective subspace so that $\mathcal{V}$ and $\mathcal{W}$ have the same dimension.
Also need to take care of deflation (linear dependencies in the Krylov spaces)!

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

$$\text{Find} \quad \arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$$

$\mathcal{H}_2$-Optimal Model Reduction

Best $\mathcal{H}_2$-norm approximation problem

Find $\arg\min_{\hat{G}\in\mathcal{H}_2 \text{ of order } \leq r} \|G - \hat{G}\|_2.$

$\rightsquigarrow$ First-order necessary $\mathcal{H}_2$-optimality conditions:

For SISO systems

$$G(-\mu_i) = \hat{G}(-\mu_i),$$
$$G'(-\mu_i) = \hat{G}'(-\mu_i),$$

where μ_i are the poles of the reduced transfer function $\hat{G}$.

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For MIMO systems

$$
\begin{aligned}
G(-\mu_i)\tilde{B}_i &= \hat{G}(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \ldots, r, \\
\tilde{C}_i^T G(-\mu_i) &= \tilde{C}_i^T \hat{G}(-\mu_i), && \text{for } i = 1, \ldots, r, \\
\tilde{C}_i^T G'(-\mu_i)\tilde{B}_i &= \tilde{C}_i^T \hat{G}'(-\mu_i)\tilde{B}_i, && \text{for } i = 1, \ldots, r,
\end{aligned}
$$

where $T^{-1}\hat{A}T = \operatorname{diag}\{\mu_1, \ldots, \mu_r\}$ = spectral decomposition and

$$\tilde{B} = \hat{B}^T T^{-T}, \quad \tilde{C} = \hat{C}T.$$

$\rightsquigarrow$ tangential interpolation conditions.

Interpolatory Model Reduction

Interpolation of the Transfer Function by Projection

Construct reduced transfer function by Petrov-Galerkin projection $\mathcal{P} = VW^T$, i.e.

$$\hat{G}(s) = CV\left(sI - W^TAV\right)^{-1}W^TB,$$

where V and W are given as the rational Krylov subspaces

$$V = \left[(-\mu_1 I - A)^{-1}B, \ldots, (-\mu_r I - A)^{-1}B\right],$$
$$W = \left[(-\mu_1 I - A^T)^{-1}C^T, \ldots, (-\mu_r I - A^T)^{-1}C^T\right].$$

Then

$$G(-\mu_i) = \hat{G}(-\mu_i) \quad \text{and} \quad G'(-\mu_i) = \hat{G}'(-\mu_i),$$

for $i = 1, \ldots, r$ as desired.

$\rightsquigarrow$ iterative algorithms (IRKA/MIRIAm) that yield $\mathcal{H}_2$-optimal models.

[Gugercin et al. '06], [Bunse-Gerstner et al. '07], [Van Dooren et al. '08]

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$\mathcal{H}_2$-Optimal Model Reduction

The basic IRKA Algorithm

Algorithm 1 IRKA

Input: A stable, B, C, $\hat{A}$ stable, $\hat{B}$, $\hat{C}$, $\delta > 0$.
Output: A^{opt}, B^{opt}, C^{opt}

1: **while** $(\max_{j=1,\ldots,r} \left\{ \frac{|\mu_j - \mu_j^{\text{old}}|}{|\mu_j|} \right\} > \delta)$ **do**
2: $\operatorname{diag}\{\mu_1, \ldots, \mu_r\} := T^{-1}\hat{A}T =$ spectral decomposition, $\tilde{B} = \hat{B}^* T^{-*}$, $\tilde{C} = \hat{C}T$.
3: $V = \left[(-\mu_1 I - A)^{-1} B\tilde{B}_1, \ldots, (-\mu_r I - A)^{-1} B\tilde{B}_r\right]$
4: $W = \left[(-\mu_1 I - A^T)^{-1} C^T\tilde{C}_1, \ldots, (-\mu_r I - A^T)^{-1} C^T\tilde{C}_r\right]$
5: $V = \operatorname{orth}(V)$, $W = \operatorname{orth}(W)$
6: $W = (W^* V)^{-1} W$
7: $\hat{A} = W^* AV$, $\hat{B} = W^* B$, $\hat{C} = CV$
8: **end while**
9: $A^{opt} = \hat{A}$, $B^{opt} = \hat{B}$, $C^{opt} = \hat{C}$