Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
Summer term 2012

Model Reduction for Dynamical Systems

— Lecture 2 —

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de

www.mpi-magdeburg.mpg.de/research/groups/csc/lehre/2012_SS_MOR/

Outline

Numerical Linear Algebra

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$\widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U\Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\|X - \widehat{X}\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \ldots, u_r, \sigma_1 v_1, \ldots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

Numerical Linear Algebra
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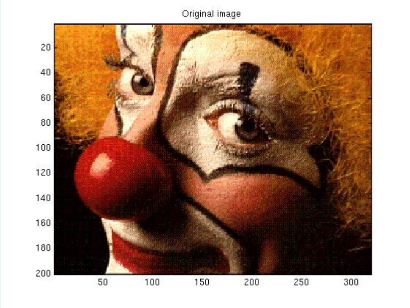
Example: Image Compression by Truncated SVD

Example: Clown



320×200 pixel

$\rightsquigarrow \quad \approx 256$ kB

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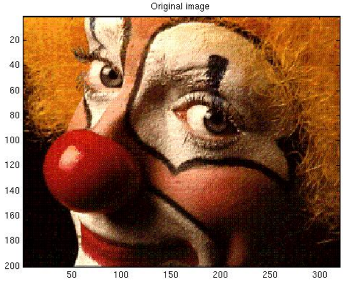

320×200 pixel
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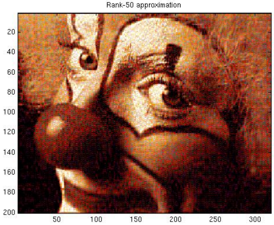
- rank $r = 50$, ≈ 104 kB



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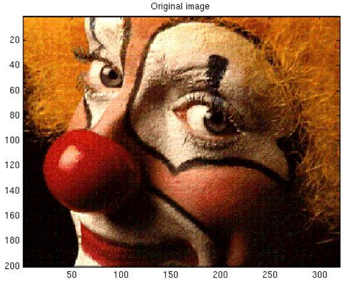

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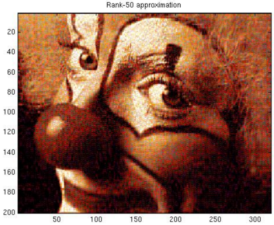
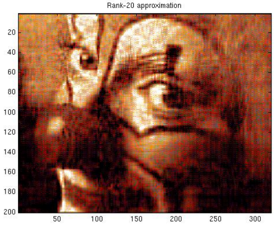


- rank $r = 20$, ≈ 42 kB



Dimension Reduction via SVD

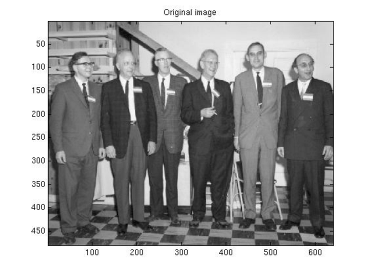
Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens,

George Forsythe, Alston Householder,

Peter Henrici, Fritz L. Bauer.



640×480 pixel, ≈ 1229 kB

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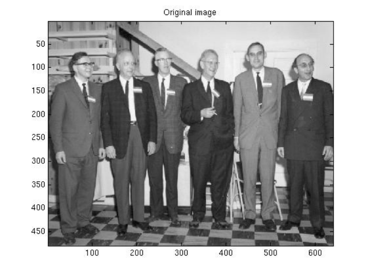
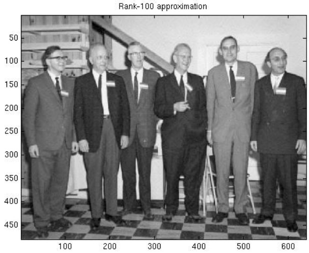
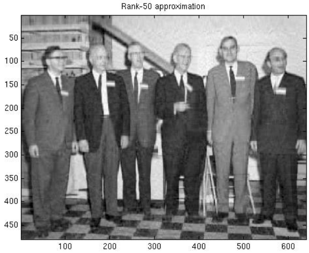


640×480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB



- rank $r = 50$, ≈ 224 kB



Background: Singular Value Decay

Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices