Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lecture 5 —

Peter Benner

Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de

www.mpi-magdeburg.mpg.de/research/groups/csc/lehre/2012_SS_MOR/

Outline

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D$$

and input functions $u \in \mathcal{L}_2^m \cong L_2^m(-\infty, \infty)$, with the 2-norm

$$\|u\|_2^2 := \frac{1}{2\pi}\int_{-\infty}^{\infty} u(\jmath\omega)^*\,u(\jmath\omega)\,d\omega.$$

Assume A (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{Re} z < 0\}$.

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Then for all $s \in \mathbb{C}^+ \cup \jmath\mathbb{R}$, $\|G(s)\| \leq M < \infty \Rightarrow$

$$\int_{-\infty}^{\infty} y(\jmath\omega)^* y(\jmath\omega)\,d\omega \quad = \quad \int_{-\infty}^{\infty} u(\jmath\omega)^* G(\jmath\omega)^* G(\jmath\omega) u(\jmath\omega)\,d\omega$$

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$$\begin{aligned} \int_{-\infty}^{\infty} y(\jmath\omega)^* y(\jmath\omega)\, d\omega &= \int_{-\infty}^{\infty} u(\jmath\omega)^* G(\jmath\omega)^* G(\jmath\omega) u(\jmath\omega)\, d\omega \\ &= \int_{-\infty}^{\infty} \|G(\jmath\omega) u(\jmath\omega)\|^2\, d\omega \leq \int_{-\infty}^{\infty} M^2 \|u(\jmath\omega)\|^2\, d\omega \end{aligned}$$

(Here, $\|\,.\,\|$ denotes the Euclidian vector or spectral matrix norm.)

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&= \quad \int_{-\infty}^{\infty} \|G(\jmath\omega)u(\jmath\omega)\|^2\,d\omega \leq \int_{-\infty}^{\infty} M^2 \|u(\jmath\omega)\|^2\,d\omega \\
&= \quad M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^* u(\jmath\omega)\,d\omega \ < \ \infty.
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&= M^2 \int_{-\infty}^{\infty} u(\jmath\omega)^* u(\jmath\omega)\, d\omega \;<\; \infty.
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$$\Longrightarrow y \in \mathcal{L}_2^p \cong L_2^p(-\infty, \infty).$$

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Assume A (asympotically) stable: $\Lambda\,(A) \subset \mathbb{C}^- := \{z \in \mathbb{C} \,:\, \operatorname{Re} z < 0\}$.
Consequently, the 2-induced operator norm

$$\|G\|_\infty := \sup_{\|u\|_2 \neq 0} \frac{\|Gu\|_2}{\|u\|_2}$$

is well defined. It can be shown that

$$\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\| = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(G(\jmath\omega)\right).$$

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Sketch of proof:
$\|G(\jmath\omega)u(\jmath\omega)\| \leq \|G(\jmath\omega)\|\|u(\jmath\omega)\| \Rightarrow$ "$\leq$".
Construct u with $\|Gu\|_2 = \sup_{\omega\in\mathbb{R}} \|G(\jmath\omega)\|\|u\|_2$.

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider transfer function

$$G(s) = C\,(sI - A)^{-1}\,B + D.$$

Hardy space $\mathcal{H}_\infty$

Function space of matrix-/scalar-valued functions that are analytic and bounded in $\mathbb{C}^+$.
The $\mathcal{H}_\infty$-norm is

$$\|F\|_\infty := \sup_{\operatorname{Re} s>0} \sigma_{\max}\left(F(s)\right) = \sup_{\omega\in\mathbb{R}} \sigma_{max}\left(F(\jmath\omega)\right).$$

Stable transfer functions are in the Hardy spaces

- $\mathcal{H}_\infty$ in the SISO case (single-input, single-output, $m = p = 1$);
- $\mathcal{H}_\infty^{p\times m}$ in the MIMO case (multi-input, multi-output, $m > 1, p > 1$).

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Paley-Wiener Theorem (Parseval's equation/Plancherel Theorem)

$$L_2(-\infty, \infty) \cong \mathcal{L}_2, \quad L_2(0, \infty) \cong \mathcal{H}_2$$

Consequently, 2-norms in time and frequency domains coincide!

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$\mathcal{H}_\infty$ approximation error

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}$.

$$\|y - \hat{y}\|_2 = \|Gu - \hat{G}u\|_2 \le \|G - \hat{G}\|_\infty \|u\|_2.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_\infty < tol$!

Note: error bound holds in time- and frequency domain due to Paley-Wiener!

Qualitative and Quantitative Study of the Approximation Error
System Norms

Consider stable transfer function

$$G(s) = C\,(sI - A)^{-1}\,B, \quad \text{i.e. } D = 0.$$

Hardy space $\mathcal{H}_2$

Function space of matrix-/scalar-valued functions that are analytic $\mathbb{C}^+$ and bounded w.r.t. the $\mathcal{H}_2$-norm

$$\begin{aligned}\|F\|_2 \quad &:= \quad \frac{1}{2\pi}\left(\sup_{\text{Re}\,\sigma>0}\int_{-\infty}^{\infty}\|F(\sigma+\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}\\ &= \quad \frac{1}{2\pi}\left(\int_{-\infty}^{\infty}\|F(\jmath\omega)\|_F^2\,d\omega\right)^{\frac{1}{2}}.\end{aligned}$$

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$\mathcal{H}_2$ approximation error for impulse response $(u(t) = u_0\delta(t))$

Reduced-order model $\Rightarrow$ transfer function $\hat{G}(s) = \hat{C}(sI_r - \hat{A})^{-1}\hat{B}$.

$$\|y - \hat{y}\|_2 = \|Gu_0\delta - \hat{G}u_0\delta\|_2 \leq \|G - \hat{G}\|_2\|u_0\|.$$

$\Longrightarrow$ compute reduced-order model such that $\|G - \hat{G}\|_2 < tol$!

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Theorem (Practical Computation of the $\mathcal{H}_2$-norm)

$$\|F\|_2^2 = \operatorname{tr}\left(B^T Q B\right) = \operatorname{tr}\left(C P C^T\right),$$

where P, Q are the controllability and observability Gramians of the corresponding LTI system.

Qualitative and Quantitative Study of the Approximation Error
Approximation Problems

$\mathcal{H}_\infty$-norm	best approximation problem for given reduced order r in general open; balanced truncation yields suboptimal solution with computable $\mathcal{H}_\infty$-norm bound.
$\mathcal{H}_2$-norm	necessary conditions for best approximation known; (local) optimizer computable with iterative rational Krylov algorithm (IRKA)
Hankel-norm $\|G\|_H := \sigma_{\max}$	optimal Hankel norm approximation (AAK theory).

Qualitative and Quantitative Study of the Approximation Error
Computable error measures

Evaluating system norms is computationally very (sometimes too) expensive.

Other measures

- absolute errors $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_2$, $\|G(\jmath\omega_j) - \hat{G}(\jmath\omega_j)\|_\infty$ $(j = 1, \ldots, N_\omega)$;
- relative errors $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_2}{\|G(\jmath\omega_j)\|_2}$, $\frac{\|G(\jmath\omega_j)-\hat{G}(\jmath\omega_j)\|_\infty}{\|G(\jmath\omega_j)\|_\infty}$;
- "eyeball norm", i.e. look at frequency response/Bode (magnitude) plot: for SISO system, log-log plot frequency vs. $|G(\jmath\omega)|$ (or $|G(\jmath\omega) - \hat{G}(\jmath\omega)|$) in decibels, 1 dB $\simeq 20\log_{10}(\text{value})$.

  For MIMO systems, $p \times m$ array of of plots G_{ij}.