Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lecture 6 —

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Outline

Model Reduction by Projection

Goals

- Automatic generation of compact models.
- Satisfy desired error tolerance for all admissible input signals, i.e., want

$$\|y - \hat{y}\| < \text{tolerance} \cdot \|u\| \qquad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

$\Longrightarrow$ Need computable error bound/estimate!

- Preserve physical properties:
  - stability (poles of G in $\mathbb{C}^-$),
  - minimum phase (zeroes of G in $\mathbb{C}^-$),
  - passivity

$$\int_{-\infty}^{t} u(\tau)^T y(\tau)\, d\tau \geq 0 \quad \forall t \in \mathbb{R}, \quad \forall u \in L_2(\mathbb{R}, \mathbb{R}^m).$$

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Model Reduction by Projection

Projection Basics

Definition 3.1 (Projector)

A projector is a matrix $P \in \mathbb{R}^{n \times n}$ with $P^2 = P$. Let $\mathcal{V} = \mathrm{range}\,(P)$, then P is projector onto $\mathcal{V}$. On the other hand, if $\{v_1, \ldots, v_r\}$ is a basis of $\mathcal{V}$ and $V = [\, v_1, \ldots, v_r\,]$, then $P = V(V^T V)^{-1} V^T$ is a projector onto $\mathcal{V}$.

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Lemma 3.2: Projector Properties

- If $P = P^T$, then P is an orthogonal projector (aka: Galerkin projection), otherwise an oblique projector (aka: Petrov-Galerkin projection).
- P is the identity operator on $\mathcal{V}$, i.e., $Pv = v \;\forall v \in \mathcal{V}$.
- $I - P$ is the complementary projector onto $\ker P$.
- If $\mathcal{V}$ is an A-invariant subspace corresponding to a subset of A's spectrum, then we call P a spectral projector.
- Let $\mathcal{W} \subset \mathbb{R}^n$ be another r-dimensional subspace and $W = [\, w_1, \ldots, w_r \,]$ be a basis matrix for $\mathcal{W}$, then $P = V(W^T V)^{-1} W^T$ is an oblique projector onto $\mathcal{V}$ along $\mathcal{W}$.

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Model Reduction by Projection

subsecname

Methods:

1. Modal Truncation
2. Rational Interpolation (Padé-Approximation and (rational) Krylov Subspace Methods)
3. Balanced Truncation
4. many more...

Joint feature of these methods:
computation of reduced-order model (ROM) by projection!

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subsecname

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Assume trajectory $x(t; u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^T x =: \tilde{x}$, where

$$\mathrm{range}\,(V) = \mathcal{V}, \quad \mathrm{range}\,(W) = \mathcal{W}, \quad W^T V = I_r.$$

Then, with $\hat{x} = W^T x$, we obtain $x \approx V\hat{x}$ so that

$$\|x - \tilde{x}\| = \|x - V\hat{x}\|,$$

and the reduced-order model is

$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

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$$\hat{A} := W^T AV, \quad \hat{B} := W^T B, \quad \hat{C} := CV, \quad (\hat{D} := D).$$

Important observations:

- The state equation residual satisfies $\dot{\tilde{x}} - A\tilde{x} - Bu \perp \mathcal{W}$, since

$$W^T \left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) \quad = \quad W^T \left(VW^T \dot{x} - AVW^T x - Bu\right)$$

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Assume trajectory $x(t;u)$ is contained in low-dimensional subspace $\mathcal{V}$. Thus, use Galerkin or Petrov-Galerkin-type projection of state-space onto $\mathcal{V}$ along complementary subspace $\mathcal{W}$: $x \approx VW^Tx =: \tilde{x}$, and the reduced-order model is $\hat{x} = W^Tx$

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$$\begin{aligned} W^T\left(\dot{\tilde{x}} - A\tilde{x} - Bu\right) &= W^T\left(VW^T\dot{x} - AVW^Tx - Bu\right) \\ &= \underbrace{W^T\dot{x}}_{\dot{\hat{x}}} - \underbrace{W^TAV}_{=\hat{A}}\underbrace{W^Tx}_{=\hat{x}} - \underbrace{W^TB}_{=\hat{B}}u \end{aligned}$$

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Model Reduction by Projection

subsecname

Projection $\rightsquigarrow$ Rational Interpolation

Given the ROM

$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

the error transfer function can be written as

$$G(s) - \hat{G}(s) \quad = \quad \left(C(sI_n - A)^{-1}B + D\right) - \left(\hat{C}(sI_r - \hat{A})^{-1}\hat{B} + \hat{D}\right)$$

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If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s)$ is a projector onto $\mathcal{V}$:

$\mathrm{range}\,(P(s)) \subset \mathrm{range}\,(V)$, all matrices have full rank $\Rightarrow$ "$=$", and

$$P(s)^2 = V(sI_r - \hat{A})^{-1}W^T(sI_n - A)V(sI_r - \hat{A})^{-1}W^T(sI_n - A)$$

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If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $P(s)$ is a projector onto $\mathcal{V}$ $\Longrightarrow$

if $(s_* I_n - A)^{-1} B \in \mathcal{V}$, *then* $(I_n - P(s_*))(s_* I_n - A)^{-1} B = 0$,

hence

$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

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$$\text{Analogously,} \quad = \quad C(sI_n - A)^{-1}\big(I_n - \underbrace{(sI_n - A)V(sI_r - \hat{A})^{-1}W^T}_{=:Q(s)}\big)B.$$

If $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, then $Q(s)^*$ is a projector onto $\mathcal{W} \Longrightarrow$

if $(s_* I_n - A)^{-*} C^T \in \mathcal{W}$, *then* $C(s_* I_n - A)^{-1}(I_n - Q(s_*)) = 0,$

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$$G(s_*) - \hat{G}(s_*) = 0 \;\Rightarrow\; G(s_*) = \hat{G}(s_*), \text{ i.e., } \hat{G} \text{ interpolates } G \text{ in } s_*!$$

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Theorem 3.3 [GRIMME '97, VILLEMAGNE/SKELTON '87]

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$$\hat{A} = W^T AV, \quad \hat{B} = W^T B, \quad \hat{C} = CV, \quad (\hat{D} = D),$$

and $s_* \in \mathbb{C} \setminus (\Lambda(A) \cup \Lambda(\hat{A}))$, if either

- $(s_* I_n - A)^{-1} B \in \mathrm{range}(V)$, or
- $(s_* I_n - A)^{-*} C^T \in \mathrm{range}(W)$,

then the interpolation condition

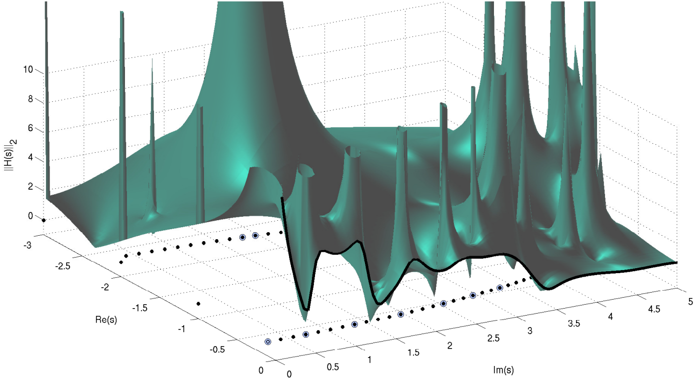
$$G(s_*) = \hat{G}(s_*).$$

in s_* holds.

Note: extension to Hermite interpolation conditions later!

Modal Truncation

Transfer Functions in $\mathbb{C}$



Modal Truncation
Example