



Model Reduction for Dynamical Systems

— Lecture 7 —

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Modal Truncation

Basic method:

Assume A is diagonalizable, $T^{-1}AT = D_A$, project state-space onto A -invariant subspace $\mathcal{V} = \text{span}(t_1, \dots, t_r)$, $t_k =$ eigenvectors corresp. to “dominant” modes / eigenvalues of A . Then with

$$V = T(:, 1:r) = [t_1, \dots, t_r], \quad \tilde{W}^* = T^{-1}(1:r,:), \quad W = \tilde{W}(V^* \tilde{W})^{-1},$$

reduced-order model is

$$\hat{A} := W^*AV = \text{diag}\{\lambda_1, \dots, \lambda_r\}, \quad \hat{B} := W^*B, \quad \hat{C} = CV$$

Also computable by truncation:

$$T^{-1}AT = \begin{bmatrix} \hat{A} & \\ & A_2 \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix}, \quad CT = [\hat{C}, C_2], \quad \hat{D} = D.$$

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Properties:

Simple computation for large-scale systems, using, e.g., Krylov subspace methods (Lanczos, Arnoldi), Jacobi-Davidson method.

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Properties:

Error bound:

$$\|G - \hat{G}\|_\infty \leq \|C_2\| \|B_2\| \frac{1}{\min_{\lambda \in \Lambda(A_2)} |\operatorname{Re}(\lambda)|}.$$

Proof:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B + D = CTT^{-1}(sI - A)^{-1}TT^{-1}B + D \\ &= CT(sI - T^{-1}AT)^{-1}T^{-1}B + D \\ &= [\hat{C}, C_2] \begin{bmatrix} (sI_r - \hat{A})^{-1} & \\ & (sI_{n-r} - A_2)^{-1} \end{bmatrix} \begin{bmatrix} \hat{B} \\ B_2 \end{bmatrix} + D \\ &= \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2, \end{aligned}$$

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Proof:

$$G(s) = \hat{G}(s) + C_2(sI_{n-r} - A_2)^{-1}B_2,$$

observing that $\|G - \hat{G}\|_\infty = \sup_{\omega \in \mathbb{R}} \sigma_{\max}(C_2(j\omega I_{n-r} - A_2)^{-1}B_2)$, and

$$C_2(j\omega I_{n-r} - A_2)^{-1}B_2 = C_2 \operatorname{diag} \left(\frac{1}{j\omega - \lambda_{r+1}}, \dots, \frac{1}{j\omega - \lambda_n} \right) B_2.$$

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Difficulties:

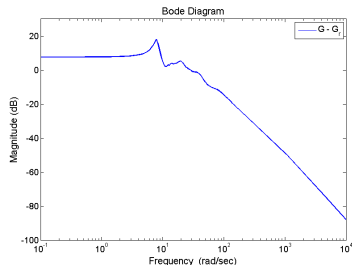
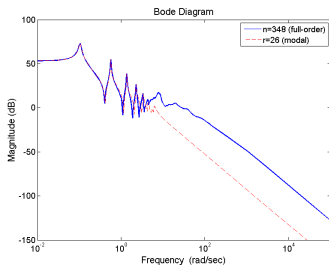
- Eigenvalues contain only limited system information.
- Dominance measures are difficult to compute.
([LITZ '79] use Jordan canonical form; otherwise merely heuristic criteria, e.g., [VARGA '95]. Recent improvement: [dominant pole algorithm](#).)
- Error bound not computable for really large-scale problems.

Modal Truncation

Example

BEAM, SISO system from [SLICOT Benchmark Collection for Model Reduction](#), $n = 348$, $m = p = 1$, reduced using 13 dominant complex conjugate eigenpairs, error bound yields $\|G - \hat{G}\|_{\infty} \leq 1.21 \cdot 10^3$

Bode plots of transfer functions and error function



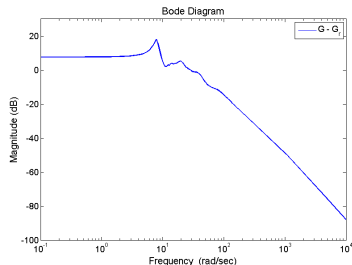
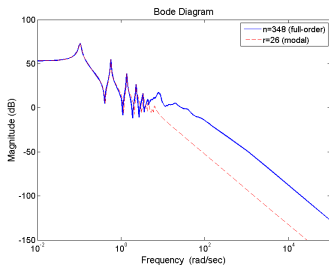
MATLAB® demo.

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MATLAB[®] demo.

Dominant Poles

Pole-Residue Form of Transfer Function

Consider partial fraction expansion of transfer function

$$G(s) = \sum_{k=1}^n \frac{R_k}{s - \lambda_k}$$

with residues $R_k := (Cx_k)(y_k^* B) \in \mathbb{C}^{p \times m}$.

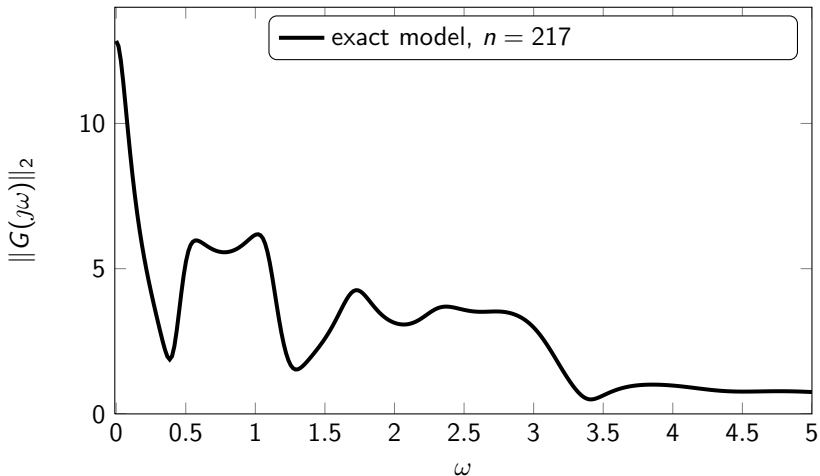
As ansatz functions, use right/left eigenvectors to **dominant poles**, i.e.. (λ_j, x_j, y_j) with largest

$$\frac{\|R_k\|}{|\Re(\lambda_k)|}$$

The dominant modes have most important influence on the input-output behavior of the system and are responsible for the "peaks" in the frequency response.

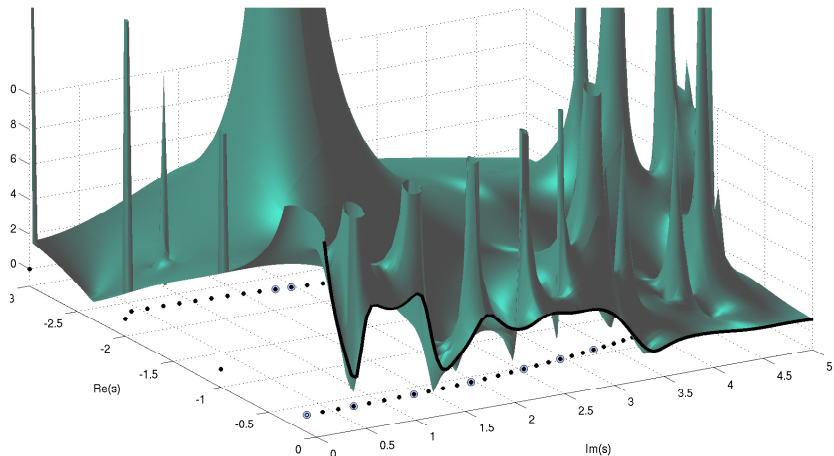
Dominant Poles

Random SISO Example ($B, C^T \in \mathbb{R}^n$)



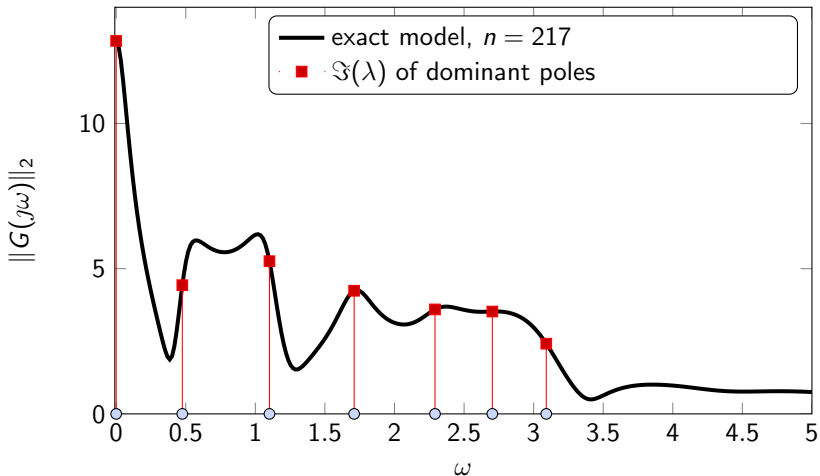
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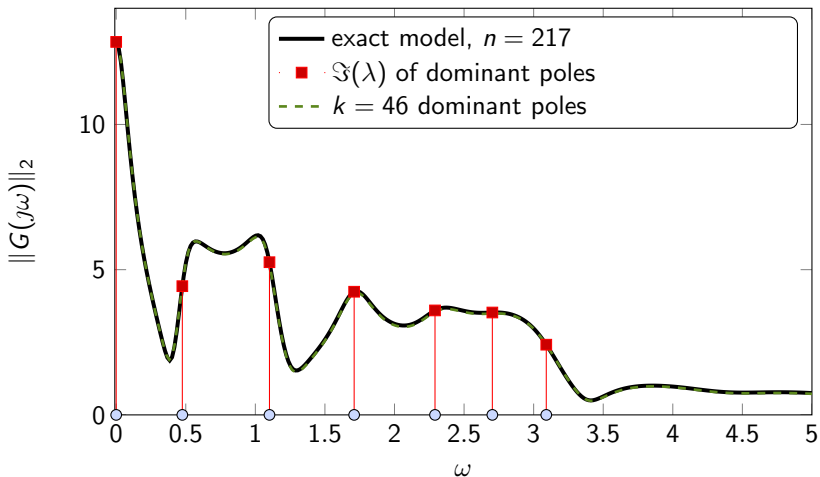
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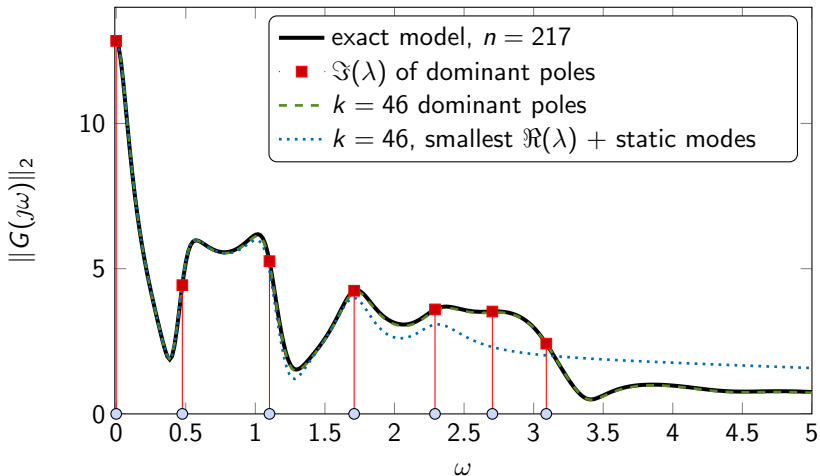
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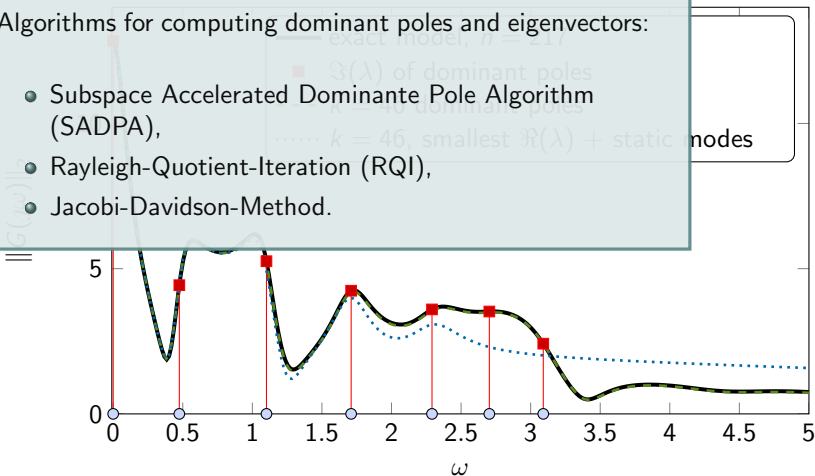


Dominant Poles

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Algorithms for computing dominant poles and eigenvectors:

- Subspace Accelerated Dominant Pole Algorithm (SADPA),
- Rayleigh-Quotient-Iteration (RQI),
- Jacobi-Davidson-Method.



Balanced Truncation

Basic principle:

- A stable system Σ , realized by (A, B, C, D) , is called **balanced**, if the **Gramians**, i.e., solutions P, Q of the **Lyapunov equations**

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \text{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$.

- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \dots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ .

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- Compute balanced realization of the system via **state-space transformation**

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

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Balanced Truncation

Motivation:

The HSVs $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \dots, \sigma_n\}$ are **system invariants**: they are preserved under

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda(PQ) = \Lambda((TPT^T)(T^{-T}QT^{-1}))$.

Balanced Truncation

Implementation: SR Method

- 1 Compute (Cholesky) factors of the Gramians, $P = S^T S$, $Q = R^T R$.
- 2 Compute SVD $SR^T = [U_1, U_2] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}$.
- 3 ROM is $(W^T A V, W^T B, C V, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \quad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

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$\implies VW^T$ is an oblique projector, hence **balanced truncation is a Petrov-Galerkin projection method.**

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \dots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum_{k=r+1}^n \sigma_k \right) \|u\|_2.$$

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