Otto-von-Guericke Universität Magdeburg
Faculty of Mathematics
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Model Reduction for Dynamical Systems

— Lecture 8 —

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Outline

Balanced Truncation

Basic principle:

- A stable system Σ, realized by (A, B, C, D), is called balanced, if the Gramians, i.e., solutions P, Q of the Lyapunov equations

$$AP + PA^T + BB^T = 0, \qquad A^T Q + QA + C^T C = 0,$$

satisfy: $P = Q = \mathrm{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n > 0$.

- $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \dots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.

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- $\Lambda\,(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the Hankel singular values (HSVs) of Σ.
- Compute balanced realization of the system via state-space transformation

$$\begin{aligned} \mathcal{T} : (A, B, C, D) &\mapsto (TAT^{-1}, TB, CT^{-1}, D) \\ &= \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix}, D \right) \end{aligned}$$

- Truncation $\rightsquigarrow (\hat{A}, \hat{B}, \hat{C}, \hat{D}) := (A_{11}, B_1, C_1, D)$.

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Motivation:

The HSVs $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are system invariants: they are preserved under

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in transformed coordinates, the Gramians satisfy

$$\begin{aligned} (TAT^{-1})(TPT^T) + (TPT^T)(TAT^{-1})^T + (TB)(TB)^T &= 0, \\ (TAT^{-1})^T(T^{-T}QT^{-1}) + (T^{-T}QT^{-1})(TAT^{-1}) + (CT^{-1})^T(CT^{-1}) &= 0 \end{aligned}$$

$$\Rightarrow \; (TPT^T)(T^{-T}QT^{-1}) = TPQT^{-1},$$

hence $\Lambda\left(PQ\right) = \Lambda\left((TPT^T)(T^{-T}QT^{-1})\right)$.

Balanced Truncation

Implementation: SR Method

1. Compute (Cholesky) factors of the Gramians, $P = S^T S, \; Q = R^T R.$
2. Compute SVD $SR^T = [\, U_1, \, U_2 \,] \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix}.$
3. ROM is $(W^T AV, W^T B, CV, D)$, where

$$W = R^T V_1 \Sigma_1^{-\frac{1}{2}}, \qquad V = S^T U_1 \Sigma_1^{-\frac{1}{2}}.$$

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$\Longrightarrow VW^T$ is an oblique projector, hence balanced truncation is a Petrov-Galerkin projection method.

Balanced Truncation

Properties:

- Reduced-order model is stable with HSVs $\sigma_1, \ldots, \sigma_r$.
- Adaptive choice of r via computable error bound:

$$\|y - \hat{y}\|_2 \leq \left(2 \sum\nolimits_{k=r+1}^{n} \sigma_k\right) \|u\|_2.$$

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Theoretical Background

Linear, Time-Invariant (LTI) Systems

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad A \in \mathbb{R}^{n\times n}, \quad B \in \mathbb{R}^{n\times m}, \\ y &= Cx + Du, \qquad C \in \mathbb{R}^{p\times n}, \quad D \in \mathbb{R}^{p\times m}. \end{aligned}$$

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Assumptions (for now): $t_0 = 0$, $x_0 = x(0) = 0$, $D = 0$.

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State-Space Description for I/O-Relation

Variation-of-constants $\Longrightarrow$

$$\mathcal{S} : u \mapsto y, \quad y(t) = \int_{-\infty}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t \in \mathbb{R}.$$

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- Recall: $A \in \mathbb{R}^{n \times m}$ is a linear operator, $A : \mathbb{R}^m \to \mathbb{R}^n$!
- Basic Idea: use SVD approximation as for matrix A!
- Problem: in general, $\mathcal{S}$ does not have a discrete SVD and can therefore not be approximated as in the matrix case!

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Instead of

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$$\mathcal{H} : u_- \mapsto y_+, \quad y_+(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu(\tau)\,d\tau \quad \text{for all } t > 0.$$

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$\rightsquigarrow$ *Hankel singular values* $\{\sigma_j\}_{j=1}^{\infty} : \ \sigma_1 \geq \sigma_2 \geq \ldots \geq 0.$

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$\Longrightarrow$ SVD-type approximation of $\mathcal{H}$ possible!

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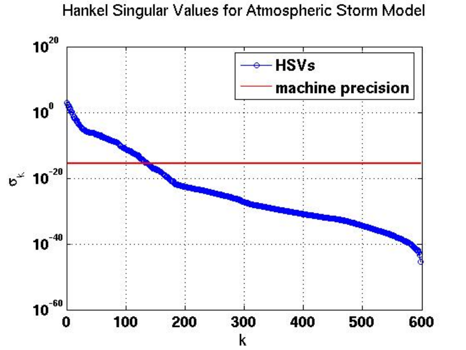

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$\Rightarrow$ solution: Adamjan-Arov-Krein (AAK Theory, 1971/78).

But: computationally unfeasible for large-scale systems.

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Theorem 4.1

Let P, Q be the controllability and observability Gramians of an LTI system Σ. Then the Hankel singular values $\Lambda(PQ)^{\frac{1}{2}} = \{\sigma_1, \ldots, \sigma_n\}$ are the singular values of the Hankel operator associated to Σ.

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Singular values of $\mathcal{H}$ = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y_+(t) = \int_0^{\infty} B^T e^{A^T(\tau-t)} C^T y_+(\tau)\,d\tau$$

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$$y_+(t) = \mathcal{H}u_-(t) = \int_{-\infty}^{0} Ce^{A(t-\tau)}Bu_-(\tau)\,d\tau = Ce^{At}z.$$

Singular values of $\mathcal{H}$ = square roots of eigenvalues of $\mathcal{H}^*\mathcal{H}$,

$$\mathcal{H}^* y_+(t) = \int_0^\infty B^T e^{A^T(\tau-t)} C^T y_+(\tau)\,d\tau = B^T e^{-A^T t} \int_0^\infty e^{A^T\tau} C^T y_+(\tau)\,d\tau.$$

Balanced Truncation

The Hankel Singular Values are Singular Values!

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$$\mathcal{H}^*y_+(t) = \; = B^Te^{-A^Tt}\int_0^\infty e^{A^T\tau}C^Ty_+(\tau)\, d\tau.$$

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$$\mathcal{H}^*\mathcal{H}u_-(t) \quad = \quad B^Te^{-A^Tt}Qz \;\doteq\; \sigma^2 u_-(t).$$

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$$\mathcal{H}^*\mathcal{H}u_-(t) = B^T e^{-A^T t} Qz \doteq \sigma^2 u_-(t).$$

$\Longrightarrow u_-(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz$

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$\Longrightarrow u_-(t) = \frac{1}{\sigma^2} B^T e^{-A^T t} Qz \Longrightarrow$ (recalling $z = \int_{-\infty}^{0} e^{-A\tau} B u_-(\tau)\, d\tau$)

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$$z \quad = \quad \int_{-\infty}^{0} e^{-A\tau} B \frac{1}{\sigma^2} B^T e^{-A^T \tau} Qz \, d\tau$$

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$\Longleftrightarrow \quad PQz = \sigma^2 z.$ □

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Theorem 4.2

Let the reduced-order system $\hat{\Sigma} : (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ with $r \leq \hat{n}$ be computed by balanced truncation. Then the reduced-order model $\hat{\Sigma}$ is balanced, stable, minimal, and its HSVs are $\sigma_1, \ldots, \sigma_r$.

Balanced Truncation

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Let the reduced-order system $\hat{\Sigma} : (\hat{A}, \hat{B}, \hat{C}, \hat{D})$ with $r \leq \hat{n}$ be computed by balanced truncation. Then the reduced-order model $\hat{\Sigma}$ is balanced, stable, minimal, and its HSVs are $\sigma_1, \ldots, \sigma_r$.

Proof: Note that in balanced coordinates, the Gramians are diagonal and equal to

$$\mathrm{diag}(\Sigma_1, \Sigma_2) = \mathrm{diag}(\sigma_1, \ldots, \sigma_r, \sigma_{r+1}, \ldots, \sigma_n).$$

Hence, the Gramian satisfies

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} + \begin{bmatrix} \Sigma_1 & \\ & \Sigma_2 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^T + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}^T = 0,$$

whence we obtain the "controllability Lyapunov equation" of the reduced-order system,

$$A_{11}\Sigma_1 + \Sigma_1 A_{11}^T + B_1 B_1^T = 0.$$

The result follows from $\hat{A} = A_{11}, \hat{B} = B_1$, $\Sigma_1 > 0$, the solution theory of Lyapunov equations and the analogous considerations for the observability Gramian. (Minimality is a simple consequence of $\hat{P} = \Sigma_1 = \hat{Q} > 0$.)