



Model Reduction for Dynamical Systems

— Lecture 9 —

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Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

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Classical Balanced Truncation (BT) [MULLIS/ROBERTS '76, MOORE '81]

- P = controllability Gramian of system given by (A, B, C, D) .
- Q = observability Gramian of system given by (A, B, C, D) .
- P, Q solve dual [Lyapunov equations](#)

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0.$$

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$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

LQG Balanced Truncation (LQGBT) [JONCKHEERE/SILVERMAN '83]

- P/Q = controllability/observability Gramian of closed-loop system based on LQG compensator.
- P, Q solve dual **algebraic Riccati equations (AREs)**

$$0 = AP + PA^T - PC^T CP + B^T B,$$

$$0 = A^T Q + QA - QB B^T Q + C^T C.$$

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Balanced Stochastic Truncation (BST) [DESAI/PAL '84, GREEN '88]

- P = controllability Gramian of system given by (A, B, C, D) , i.e., solution of **Lyapunov equation** $AP + PA^T + BB^T = 0$.
- Q = observability Gramian of right spectral factor of power spectrum of system given by (A, B, C, D) , i.e., solution of **ARE**

$$\hat{A}^T Q + Q \hat{A} + QB_W(DD^T)^{-1}B_W^T Q + C^T(DD^T)^{-1}C = 0,$$

where $\hat{A} := A - B_W(DD^T)^{-1}C$, $B_W := BD^T + PC^T$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Positive-Real Balanced Truncation (PRBT)

[GREEN '88]

- Based on positive-real equations, related to positive real (Kalman-Yakubovich-Popov-Anderson) lemma.
- P, Q solve dual **AREs**

$$0 = \bar{A}P + P\bar{A}^T + PC^T\bar{R}^{-1}CP + B\bar{R}^{-1}B^T,$$

$$0 = \bar{A}^TQ + Q\bar{A} + QB\bar{R}^{-1}B^TQ + C^T\bar{R}^{-1}C,$$

where $\bar{R} = D + D^T$, $\bar{A} = A - B\bar{R}^{-1}C$.

Balancing-Related Methods

Basic Principle

Given positive semidefinite matrices $P = S^T S$, $Q = R^T R$, compute balancing state-space transformation so that

$$P = Q = \text{diag}(\sigma_1, \dots, \sigma_n) = \Sigma, \quad \sigma_1 \geq \dots \geq \sigma_n > 0,$$

and truncate corresponding realization at size r with $\sigma_r > \sigma_{r+1}$.

Other Balancing-Based Methods

- Bounded-real balanced truncation (BRBT) – based on bounded real lemma [OPDENACKER/JONCKHEERE '88];
- H_∞ balanced truncation (HinfBT) – closed-loop balancing based on H_∞ compensator [MUSTAFA/GLOVER '91].

Both approaches require solution of dual AREs.

- Frequency-weighted versions of the above approaches.

Balancing-Related Methods

Properties

- Guaranteed preservation of physical properties like
 - stability (all),
 - passivity (PRBT),
 - minimum phase (BST).
- Computable error bounds, e.g.,

$$\text{BT: } \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^n \sigma_j^{BT},$$

$$\text{LQGBT: } \|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^n \frac{\sigma_j^{LQG}}{\sqrt{1+(\sigma_j^{LQG})^2}}$$

$$\text{BST: } \|G - G_r\|_\infty \leq \left(\prod_{j=r+1}^n \frac{1+\sigma_j^{BST}}{1-\sigma_j^{BST}} - 1 \right) \|G\|_\infty,$$

- Can be combined with singular perturbation approximation for steady-state performance.
- Computations can be modularized.