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Computational Methods for Systems and Control Theory
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Linear Systems and Matrix Equations – Question sheet 1.

There will be no work to hand in put you will be asked to perform one of the questions on the board.

Problem 1 **(2 points)**
Familiarize yourself with the first chapter in the book "Iterative methods for sparse linear systems" by Yousef Saad. `http://www-users.cs.umn.edu/~saad/IterMethBook_2ndEd.pdf`

Problem 2 **(4 points)**
Let A be a 4×4 matrix to which the following operations are applied

1. double column 2,
2. halve row 1,
3. add row 2 to row 4,
4. interchange columns 1 and 2,
5. substract row 1 from each of the other rows,
6. replace column 4 by column 3,
7. delete column 2 (so that the column dimension is reduced by 1).

(a) Write this as the product of 8 matrices.

(b) Write this as the product of three matrices UAV with A the original matrix.

Problem 3 **(4 points)**
An important role in numerical linear algebra is played by matrices used for similarity transformations of the original matrix A to obtain a matrix SAS^{-1} with more desirable properties. Givens roations defined as matrices

$$G(i,k,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

where $c = cos(\theta)$ and $s = sin(\theta)$. These matrices are used to eliminate entries from vectors and matrices. Derive expressions for c and s based on the fact that this matrix is used to eliminate the entry in position k of the vector x using the vector element in position j, i.e., $y := G(i,k,\theta)x$ with $y_k = 0$. Using this show that the matrix G is orthogonal.

Problem 4 **(2 points)**

Show that for any symmetric positive definite matrix $A \in \mathbb{R}^{n,n}$ (see 1.11 in the book from Question 1) all diagonal entries must be positive.

Problem 5 **(6 points)**

(a) Let A be an $n \times n$ matrix, with entries a_{ij}. For $i \in \{1, \ldots, n\}$ let $R_i = \sum_{j \neq i} |a_{ij}|$ be the sum of the absolute values of the non-diagonal entries in the ith row. Let $D(a_{ii}, R_i)$ be the closed disc centered at a_{ii} with radius R_i. Prove that every eigenvalue of A lies within at least one of the discs $D(a_{ii}, R_i)$. [Hint: Start using the eigenvalue relation $Ax = \lambda x$.]

(b) Show using the previous result that the matrix

$$A = \begin{bmatrix} 5 & 1 & 0 & 0 & 0 & 0 \\ 1 & 10 & 1 & 0 & 0 & 0 \\ 0 & -1 & 15 & 1 & 0 & 0 \\ 0 & 0 & -1 & 20 & 1 & 0 \\ 0 & 0 & 0 & -1 & 25 & 1 \\ 0 & 0 & 0 & 0 & -1 & 30 \end{bmatrix}$$

is invertible.

(c) Assuming that *Theorem: If the union of k discs is disjoint from the union of the other $n-k$ discs then the former union contains exactly k and the latter $n-k$ eigenvalues of A* holds, show that the matrix A from part (b) has only real eigenvalues.

Problem 6 **(2 points)**

Let A be an $n \times n$ square matrix. Show using Gaussian elimination that $Ax = b$ has a unique solution iff (if and only if) the diagonal entries u_{jj} of U from the LU factorization of $A = LU$ are nonzero. (Hint: Consider determinants).

Problem 7 **(4 points)**

Suppose that the $n \times n$ matrix A is written in block-form, i.e.,

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{m,m}$ and $A_{22} \in \mathbb{R}^{n-m,n-m}$ as well as A_{11} and A being invertible.

(a) Verify that

$$\begin{bmatrix} I & 0 \\ -A_{21}A_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} - A_{21}A_{11}^{-1}A_{12} \end{bmatrix}.$$

Here $A_{22} - A_{21}A_{11}^{-1}A_{12}$ is called the Schur-complement of A.

(b) What are the conditions on the Schur-complement $S = A_{22} - A_{21}A_{11}^{-1}A_{12}$ (provided A_{11}^{-1} exists) for A to be invertible?