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Otto-von-Guericke-University Magdeburg
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods for Systems and Control Theory

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Scientific Computing 1
Handout 7
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Error Concepts, Stability and Conditioning

Consider $y = f(x)$, $x \in D \subset \mathbb{R}^n$, $y \in V \subset \mathbb{R}^m$ and the computed result $\hat{y} = y + \Delta y = f(x + \Delta x)$

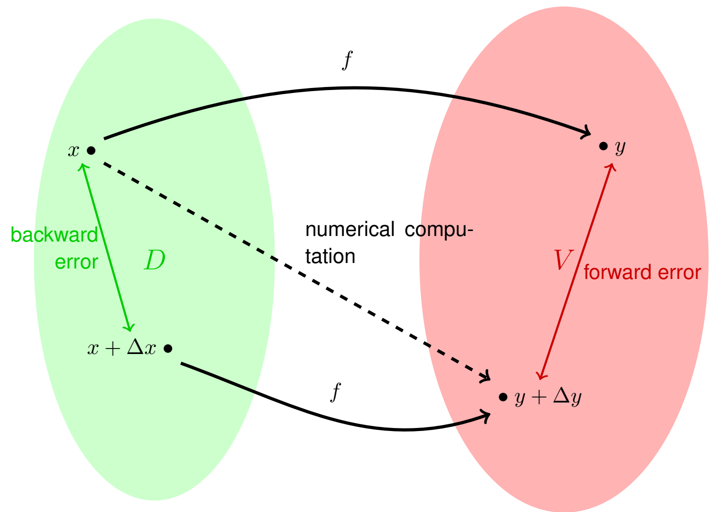

Figure 1: relation of forward and backward errors and numerical computation

- **forward error:**
  - *absolute:* $\|y - \hat{y}\|$,
  - *relative:* $\|y - \hat{y}\| / \|y\|$.

  Estimated in a *forward error analysis*.

- **backward error:**
  - *absolute:* η,
  - *relative:* $\eta / \|x\|$,

  where $\eta := \inf\{\|\Delta x\| \,|\, \hat{y} = f(x + \Delta x)\}$. Estimated in a *backward error analysis*.

- **numerical stability** (of an algorithm):
  - An algorithm producing a relative backward error of the magnitude of the relative data errors $\frac{\|\Delta x\|}{\|x\|}$ is called *(numerically) backward stable*.
  - If an algorithm produces relative forward errors of the magnitude a backward stable algorithm would produce, then it is denoted *(numerically) forward stable*.
- **backward stable $\Rightarrow$ forward stable**
- **(relative) condition number:**

  $$c(f,x) := \frac{\|x\|}{\|f(x)\|}\|f'(x)\|$$

  Depends on problem *and* the data. Mathematical problems are often not generally badly conditioned but conditionally badly conditioned depending on the data.

  - $c(f,x) \approx 1 \Rightarrow$ well conditioned
  - $c(f,x) \gg 1 \Rightarrow$ badly conditioned
  - $c(f,x) \ll 1 \Rightarrow$ possibly bad due to “loss of information” $leftrightarrow$ large backward error.

  An algorithm can already become unstable if a (well conditioned) problem is subdivided in several steps and only one step is ill conditioned.
- **rule of thumb:**

  forward error $\lessapprox$ condition number $\times$ backward error

- **justification of computational results:**
  - good conditioning and stable algorithm $\Rightarrow$ reliable results
  - bad conditioning or unstable algorithm $\Rightarrow$ unsure results