Chapter 3

Multicore and Multiprocessor Systems: Part IV

Tree Reduction

The OpenMP reduction minimal example revisited: Data Sharing

Example (OpenMP reduction minimal example)

```
#include <omp.h>
#include <stdio.h>
#include <stdlib.h>

int main (int argc, char *argv[]) {
int   i, n;
float a[100], b[100], sum;

/* Some initializations */
n = 100;
for (i=0; i < n; i++)
  a[i] = b[i] = i * 1.0;
sum = 0.0;

#pragma omp parallel for reduction(+:sum)
  for (i=0; i < n; i++)
    sum = sum + (a[i] * b[i]);
printf("   Sum = %f\n",sum);
}
```

Tree Reduction

The OpenMP reduction minimal example revisited

The main properties of the reduction are

- accumulation of data via a binary operator (here $+$)
- intrinsically sequential operation causing a race condition in multi-thread based implementations (since every iteration step depend on the result of its predecessor.)

Tree Reduction

Basic idea of tree reduction

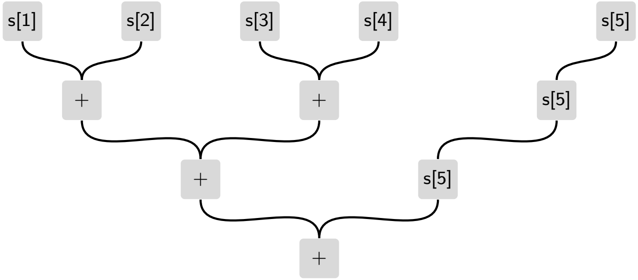

Figure: Tree reduction basic idea.

Tree Reduction

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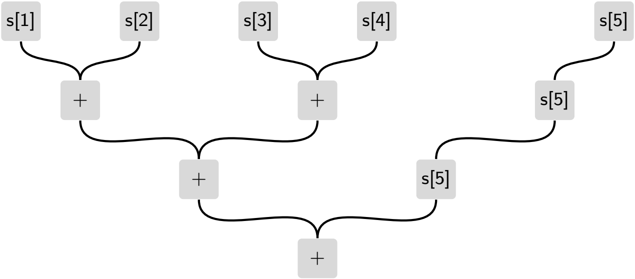

Figure: Tree reduction basic idea.

- ideally the number of elements is a power of 2
- best splitting of the actual data depends on the hardware used

Tree Reduction

Practical tree reduction on multiple cores

Example (Another approach for the `dot` example)

Consider the setting as before $a, b \in \mathbb{R}^{100}$. Further we have four equal cores. How do we compute the accumulation in parallel?

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Consider the setting as before $a, b \in \mathbb{R}^{100}$. Further we have four equal cores. How do we compute the accumulation in parallel? Basically 2 choices

1. **Task pool approach:** define a task pool and feed it with $n/2 = 50$ work packages accumulating 2 elements in 1. When these are done, schedule the next 25 and so on by further binary accumulation of 2 intermediate results per work package.

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1. **Task pool approach:** define a task pool and feed it with $n/2 = 50$ work packages accumulating 2 elements in 1. When these are done, schedule the next 25 and so on by further binary accumulation of 2 intermediate results per work package.
2. **#Processors=#Threads approach:** Divide the work by the number of threads, i.e. on our 4 cores each gets 25 subsequent indices to sum up. The reduction is then performed on the results of the threads.

Dense Linear Systems of Equations

Repetition blocked algorithms

Algorithm 1: Gaussian elimination – row-by-row-version

Input: $A \in \mathbb{R}^{n\times n}$ allowing LU decomposition
Output: A overwritten by L, U

```
for k = 1 : n − 1 do
    ;
    ;
    A(k + 1 : n, k) = A(k + 1 : n, b)/A(k, k);
    for i = k + 1 : n do
        for j = k + 1 : n do
            A(i, j) = A(i, j) − A(i, k)A(k, j);
```

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Observation:

- Innermost loop performs rank-1 update on the $A(k+1:n, k+1:n)$ submatrix in the lower right,
- i.e. a BLAS level 2 operation.

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Algorithm 2: Gaussian elimination – Outer product formulation

Input: $A \in \mathbb{R}^{n \times n}$ allowing LU decomposition
Output: $L, U \in \mathbb{R}^{n \times n}$ such that $A = LU$ stored in A stored in A

```
for k = 1 : n − 1 do
    rows= k + 1 : n;
    A(rows, k) = A(rows, k)/A(k, k);
    A(rows,rows) = A(rows,rows) − A(rows, k)A(k,rows);
```

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Idea of the blocked version

- Replace the rank-1 update by a rank-r update ,
- Thus replace the $\mathcal{O}(n^2)$ / $\mathcal{O}(n^2)$ operation per data ratio the more desirable $\mathcal{O}(n^3)$ / $\mathcal{O}(n^2)$ ratio,
- Therefore exploit the fast local caches of modern CPUs more optimally.

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Repetition blocked algorithms

Algorithm 3: Gaussian elimination – Block outer product formulation

Input: $A \in \mathbb{R}^{n \times n}$ allowing LU decomposition, r prescribed block size

Output: $A = LU$ with L, U stored in A

```
k = 1;
while k ≤ n do
    ℓ = min(n, k + r − 1);
    Compute A(k : ℓ, k : ℓ) = L̃Ũ via Algorithm 7;
    Solve L̃Z = A(k : ℓ, ℓ + 1 : n) and store Z in A;
    Solve WŨ = A(ℓ + 1 : n, k : ℓ) and store W in A;
    Perform the rank-r update:
    A(ℓ + 1 : n, ℓ + 1 : n) = A(ℓ + 1 : n, ℓ + 1 : n) − WZ;
    k = ℓ + 1;
```

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$k = 1$;
while $k \leq n$ **do**
  $\ell = \min(n, k + r - 1)$;
  Compute $A(k : \ell, k : \ell) = \tilde{L}\tilde{U}$ via Algorithm 7;
  Solve $\tilde{L}Z = A(k : \ell, \ell + 1 : n)$ and store Z in A;
  Solve $W\tilde{U} = A(\ell + 1 : n, k : \ell)$ and store W in A;
  Perform the rank-r update:
  $A(\ell + 1 : n, \ell + 1 : n) = A(\ell + 1 : n, \ell + 1 : n) - WZ$;
  $k = \ell + 1$;

The block size r can be further exploited in the computation of W and Z and the rank-r update. It is used to optimize the data portions for the cache.

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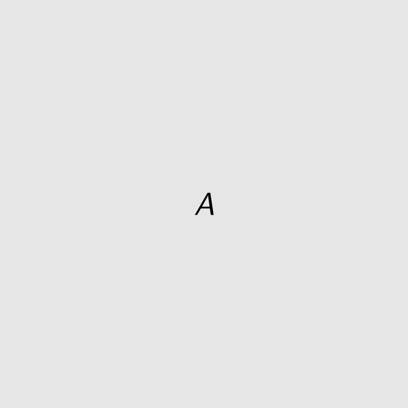
Repetition blocked algorithms



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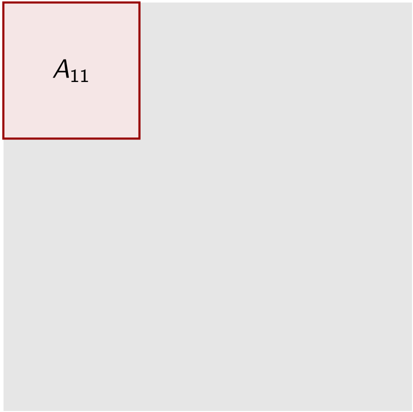
Repetition blocked algorithms



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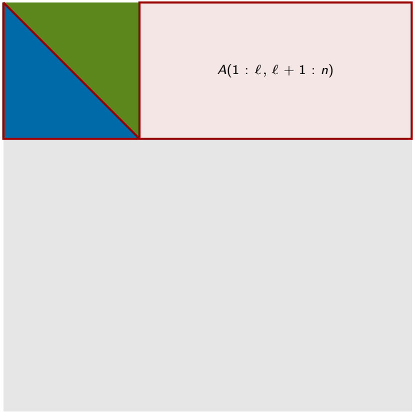

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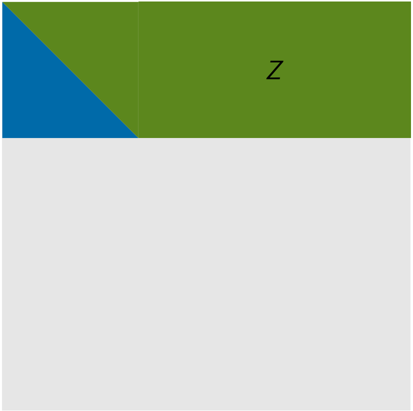
Repetition blocked algorithms



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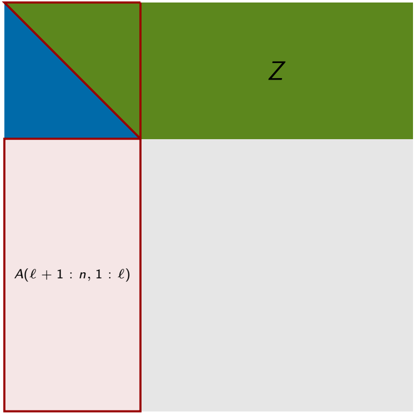

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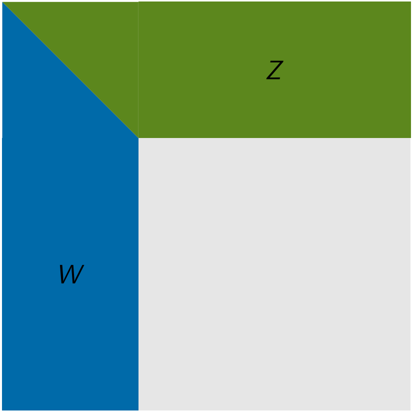

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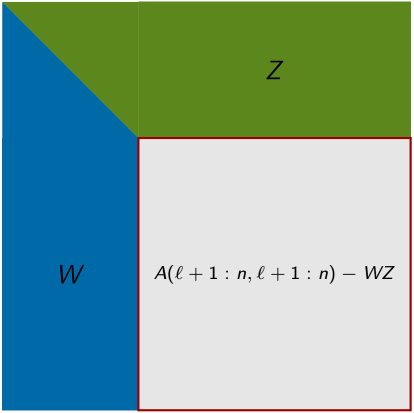

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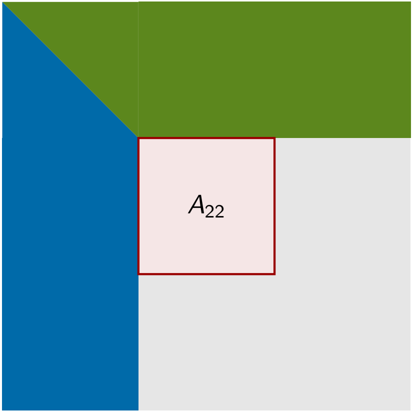

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Fork-Join parallel implementation for multicore machines

We have basically two ways to implement naive parallel versions of the block outer product elimination in Algorithm 6.

Threaded BLAS available

- Compute line 4 with the sequential version of the LU
- Exploite the threaded BLAS for the block operations in lines 5–7

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Netlib BLAS

- Compute line 4 with the sequential version of the LU
- Employ OpenMP/PThreads to perform the BLAS calls for the block operations in lines 5–7 in parallel.

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Fork-Join parallel implementation for multicore machines

Both these approaches fall into the class of parallel codes described by the following definition.

Definition (Fork-Join Parallelism)

An algorithm that performs a loop where certain parts need to be performed in a sequential way, but others can be done in parallel is *called fork-join-parallel*.

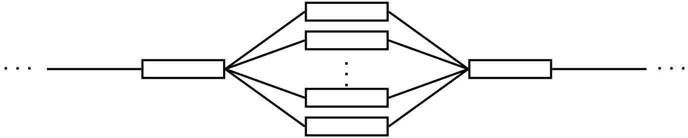

Figure: A sketch of the fork-join execution model.

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Fork-Join parallel implementation for multicore machines

Advantages

- Easy to achieve,
- many threaded BLAS implementations available,
- Basically usable from any user code that requires linear system solves

Disadvantages

- very naive implementation,
- sequential fraction limits the speedup (Amdahl's law),
- i.e., only useful for small numbers of cores.

Dense Linear Systems of Equations

DAG scheduling of block operations aiming at manycore systems

Definition (Directed Acyclic Graph (DAG))

A *directed acyclic graph* is a graph where

- all edges have one distinct direction,
- directions are such that no cycles are possible for any path in the graph.

Where is the connection to parallel mathematical algorithms?

- Consider every node in the graph a task in the computation.

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- Also none of the previous tasks depend on the later ones.

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- Every task requires certain number of previous tasks to have finished.
- Also none of the previous tasks depend on the later ones.
- Thus, the dependencies give us the directions and cycles can not appear by construction.

Dense Linear Systems of Equations

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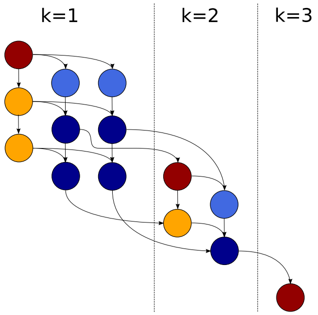

Figure: Dependency graph of Algorithm 6 for a 3×3 block subdivision.

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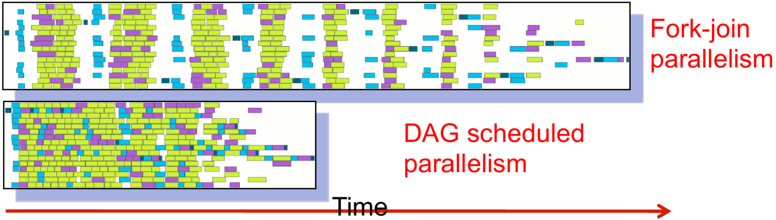

Figure: The superiority of DAG scheduling of tasks over fork-join parallelism.