

Scientific Computing 1
Handout 9
January 21, 2015

Projection Methods and Conjugate Gradients

• **Projection Method:**

- Let $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, K_m, L_m m -dimensional subspaces of \mathbb{R}^n . $V_m, W_m \in \mathbb{R}^{n \times m}$ with full column rank, containing bases of K_m, L_m respectively. $x_0 \in \mathbb{R}^n$ an initial vector.
- A **projection method** for $Ax = b$ searches for a solution (approximation) $x_m \in x_0 + K_m$ that satisfies

$$b - Ax_m \perp L_m. \quad (1)$$

- If $K_m = L_m$ (1) is called **(Ritz-)Galerkin condition** and the method is called **orthogonal projection method**.
- If $K_m \neq L_m$ (1) is called **Petrov-Galerkin condition** and the method is called **oblique projection method**.

• **Krylov Subspaces and Krylov Subspace Methods:**

Let $y \in \mathbb{R}^n$ be an arbitrary vector.

$$K_m(A, y) = \text{span}\{y, Ay, \dots, A^{m-1}y\}$$

is called the m -th Krylov subspace for A and y . A projection method with $K_m = K_m(A, y)$ is called **Krylov subspace (projection) method**.

• A prototype Krylov Subspace Method:

Algorithmus 1 Conjugate Gradient Method

Input: $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$

Output: $x = A^{-1}b$

$$p_0 = r_0 = b - Ax_0, \alpha_0 = \|r_0\|^2$$

for $m := 0, \dots, n - 1$ **do**

if $\alpha_m \neq 0$ **then**

$$v_m = Ap_m$$

$$\lambda_m = \frac{\alpha_m}{(v_m, p_m)}$$

$$x_{m+1} = x_m + \lambda_m p_m$$

$$r_{m+1} = r_m - \lambda_m v_m$$

$$\alpha_{m+1} = \|r_{m+1}\|^2$$

$$p_{m+1} = r_{m+1} + \frac{\alpha_{m+1}}{\alpha_m} p_m$$

else

 STOP

end if

end for

Algorithmus 2 Preconditioned Conjugate Gradient Method

Input: $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $x_0 \in \mathbb{R}^n$, $A^{-1} \approx P \in \mathbb{R}^{n \times n}$

Output: $x = A^{-1}b$

$$r_0 = b - Ax_0, p_0 = Pr_0, \alpha_0 = (r_0, p_0)$$

for $m := 0, \dots, n - 1$ **do**

if $\alpha_m \neq 0$ **then**

$$v_m = Ap_m$$

$$\lambda_m = \frac{\alpha_m}{(v_m, p_m)_2}$$

$$x_{m+1} = x_m + \lambda_m p_m$$

$$r_{m+1} = r_m - \lambda_m v_m$$

$$z_{m+1} = Pr_{m+1}$$

$$\alpha_{m+1} = (r_{m+1}, z_{m+1})_2$$

$$p_{m+1} = z_{m+1} + \frac{\alpha_{m+1}}{\alpha_m} p_m$$

else

 STOP

end if

end for
