Otto-von-Guericke-University Magdeburg
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory

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Winterterm 2016/2017

Scientific Computing 1
7th Homework

Handout: 12/01/2016 **Return:** 12/09/2016

Exercise 1: **(2 Points)**

Let V be an n-dimensional pre-Hilbert space over $\mathbb{R}$. The inner product is defined as

$$\langle x, y \rangle := x^T y.$$

Show that this is a proper definition of an inner product and prove that

$$||x|| = \sqrt{\langle x, x \rangle}$$

defines a norm on this space.

Exercise 2: **(2 Points)**

Prove that the linear system $Ax = b$ is solvable if and only if

$$\operatorname{rank}(A) = \operatorname{rank}([A\ b])$$

Does this condition guarantee uniqueness?

Exercise 3: **(2 Points)**

Let $L, \tilde{L} \in \mathbb{R}^{n \times n}$ be two lower triangular matrices. Show that the product $T = L\tilde{L}$ also is a lower triangular matrix.

Exercise 4: **(3 Points)**

a.) Let $Q, \tilde{Q} \in \mathbb{C}^{n \times n}$ be two unitary matrices. Show that the product $Q\tilde{Q}$ is also an unitary matrix.

b.) The Givens rotation is defined as

$$G(i,j,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \cdots & c & \cdots & -s & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & s & \cdots & c & \cdots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

where $c = \cos(\theta)$ and $s = sin(\theta)$ appear at the intersections of the i-th and j-th rows and columns. Prove that $G(i, j, \theta)$ is a unitary matrix for arbitrary parameters.

Exercise 5: **(3 Points)**

Let $I \in \mathbb{R}^{n\times n}$ be the identity matrix.

a.) Show that $||I|| = 1$ holds for all induced matrix norms.

b.) Why is the Frobenius norm $||\cdot||_F$ not an induced matrix norm?

Exercise 6: **(2 Points)**

The 2-norm of a matrix $A \in \mathbb{C}^{n\times n}$ can be computed as

$$||A||_2 = \sqrt{\varrho(A^H A)}.$$

Why is it reasonable to avoid the computation of the matrix 2-norm?

Exercise 7: **(6 Points)**

Let $x \in \mathbb{R}^n$. Prove the following inequalities:

a.) $||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$

b.) $||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$

c.) $\frac{1}{n}||x||_1 \leq ||x||_\infty \leq ||x||_1$

Use the results to show the corresponding inequalities for a matrix $A \in \mathbb{C}^{n\times n}$. Finally, show that all norms are finite.

Exercise 8: **(6 Points)**

a.) Prove the Schur Decomposition:

For all $A \in \mathbb{C}^{n\times n}$ there exists a unitary matrix $U \in \mathbb{C}^{n\times n}$ and an upper triangular matrix $T \in \mathbb{C}^{n\times n}$ such that

$$A = U^H T U$$

holds with $\operatorname{diag}(T) = \{\lambda_1, \ldots, \lambda_n\}$.
Hint: Use an induction over the matrix dimension.

b.) Show that the eigenvalues λ_i can be arranged in arbitrary order on the diagonal using Givens rotations. Show that this can be used to compute the eigenvector v_i belonging to a desired eigenvalue λ_i.

Overall Points: 26