Otto von Guericke University Magdeburg
Institute for Analysis and Numerics
Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory

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Model Reduction of Dynamical Systems
Homework 1

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Deadline: 5 May 2017

Problem 1. *Properties of the Singular Value Decomposition (SVD)*
Let $A = U\Sigma V^T \in \mathbb{R}^{m\times n}$ be the singular value decomposition with orthogonal matrices $U = \begin{bmatrix} u_1 & \dots & u_m \end{bmatrix} \in \mathbb{R}^{m\times m}$ and $V = \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} \in \mathbb{R}^{n\times n}$, and a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ with $\sigma_1 \geq \dots \geq \sigma_{\min\{m,n\}} \geq 0$ on the diagonal. Additionally, let $r = \operatorname{rank}(A) < \min\{m, n\}$, i.e. $\sigma_r > 0$ and $\sigma_{r+1} = 0$.

(a) Prove that

$$\ker(A) = \operatorname{span}\{v_{r+1}, \dots, v_n\}\,,$$
$$\operatorname{range}(A) = \operatorname{span}\{u_1, \dots, u_r\}\,.$$

(b) Prove that

$$\|A\|_F = \sqrt{\sigma_1^2 + \ldots + \sigma_r^2}.$$

(c) What are the eigenvalues and eigenvectors of A^TA and AA^T?

(d) What are the singular values of a normal matrix A with eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$?

(e) (Schmidt-Eckart-Young-Mirsky theorem). Let $k < r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$. Prove that

$$\min_{\operatorname{rank}(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}.$$

Hint: Show that, for a matrix $B \in \mathbb{R}^{m\times n}$ with $\operatorname{rank}(B) = k$, there exists $\|z\|_2 = 1$ such that $Bz = 0$ and $z \in \operatorname{span}\{v_1, \dots, v_{k+1}\}$. Then show that $\|A - B\|_2 \geq \|Az - Bz\|_2$ and $\|Az - Bz\|_2 = \|Az\|_2 \geq \sigma_{k+1}$.

(f) What is the distance (in terms of the spectral norm) of a square invertible matrix $A \in \mathbb{R}^{n\times n}$ (with singular values $\sigma_1, \dots, \sigma_n$) to the set of singular matrices?

Problem 2. *Controllability of LTI systems*
Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, and $\mathcal{C} = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ the associated Kalman controllability matrix.

(a) Show that the controllability subspace $\operatorname{range}(\mathcal{C})$ is an A-invariant subspace, i.e. $A \operatorname{range}(\mathcal{C}) \subseteq \operatorname{range}(\mathcal{C})$.

(b) Let $T \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that the controllability matrix of $(T^{-1}AT, T^{-1}B)$ is $T^{-1}\mathcal{C}$.

(c) (Kalman controllability form). Let $r = \text{rank}(\mathcal{C})$, $0 < r < n$. Show that there exists an invertible matrix $T \in \mathbb{R}^{n\times n}$ such that

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \text{ and } \tilde{B} := T^{-1}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix},$$

with $A_{11} \in \mathbb{R}^{r\times r}$ and $B_1 \in \mathbb{R}^{r\times m}$, where (A_{11}, B_1) is a controllable matrix pair.

(d) Find a Kalman controllability form for the pair

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(e) (Popov-Belevich-Hautus (PBH) test). Show that (A, B) is controllable if and only if the matrix $\begin{bmatrix} \lambda I - A & B \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.

Problem 3. *Properties of LTI systems*
Consider the dynamical system given by

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Analyze the above system with respect to stability, controllability, and observability. Further, compute the transfer function $H(s) = C(sI - A)^{-1}B$ of the system. What can you say with regard to minimality of the system?

Problem 4. *Minimality of LTI systems*

(a) Show that realizations (A, B, C, D) and $(T^{-1}AT, T^{-1}B, CT, D)$ are equivalent, i.e. they have the same transfer function.

(b) Prove that two realizations are equivalent if and only if their Markov parameters are equal. More precisely, given two realizations (A_i, B_i, C_i, D_i), $i = 1, 2$, prove that

$$C_1(sI - A_1)^{-1}B_1 + D_1 = C_2(sI - A_2)^{-1}B_2 + D_2$$

if and only if

$$D_1 = D_2 \text{ and } C_1 A_1^k B_1 = C_2 A_2^k B_2, \text{ for } k \geq 0.$$

(c) Show that if (A, B) is not controllable, then the realization (A, B, C, D) is not minimal.

Hint: Apply the Kalman controllability decomposition.

(d) Show that if (A, C) is not observable, then the realization (A, B, C, D) is not minimal.

Hint: Apply the Kalman observability decomposition

$$\tilde{A} := T^{-1}AT = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \text{ and } \tilde{C} := CT = \begin{bmatrix} C_1 & 0 \end{bmatrix},$$

which yields an observable matrix pair (A_{11}, C_1).

(e) Show that if (A, B) is controllable and (A, C) is observable, then the realization (A, B, C, D) is minimal.

Hint: Consider the Hankel matrix (the product of the observability and controllability matrix).

(f) Prove that if (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) are equivalent and minimal representations, then there exists an invertible matrix T such that

$$A_2 = T^{-1}A_1T, \quad B_2 = T^{-1}B_1, \quad C_2 = C_1T, \quad D_2 = D_1.$$

Problem 5. *Lyapunov equation*
Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$. Show that, if any two of the following three statements are true, then all of them are true:

(a) A is Hurwitz,

(b) (A, B) is controllable,

(c) there exists $P > 0$ such that $AP + PA^T + BB^T = 0$.

Problem 6. *Computation of system norms*

(a) Consider the LTI system

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}, \quad D = 0.$$

Analytically compute the $\mathcal{H}_\infty$-norm of the system.

(b) Consider the following LTI system:

$$A = \begin{bmatrix} -8 & 8 \\ -8 & -42 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 2 \end{bmatrix}, \quad D = 0.$$

Compute the $\mathcal{H}_2$-norm of the system by first solving a Lyapunov equation.

Problem 7. *Balanced realizations*
Let (A, B, C, D) be a minimal realization of an LTI system. Show that a balanced realization is given by the state-space transformation

$$T := \Sigma^{-\frac{1}{2}} V^T R,$$

where $P = S^T S$ and $Q = R^T R$ (e.g., Cholesky decompositions) satisfy the pair of Lyapunov equations

$$AP + PA^T + BB^T = 0,$$
$$A^T Q + QA + C^T C = 0$$

and

$$SR^T = U \Sigma V^T$$

is the SVD of SR^T.

Hint: First note that $T^{-1} = S^T U \Sigma^{-\frac{1}{2}}$, then the result follows by simple algebraic manipulations.

You can send your solutions in two ways:

1. **by sending them to `mlinaric@mpi-magdeburg.mpg.de`,**
2. **by adding Petar Mlinarić (username `pmli`) to your GitLab or Bitbucket project.**

Solutions should be written in a PDF file (created using LaTeX) or a Jupyter notebook. When sending emails, please add `[mor17]` to the subject line.