MAX PLANCK INSTITUTE
FOR DYNAMICS OF COMPLEX
TECHNICAL SYSTEMS
MAGDEBURG

CSC COMPUTATIONAL METHODS IN
SYSTEMS AND CONTROL THEORY

Model Reduction for Dynamical Systems

–Lecture 2–

Peter Benner **Lihong Feng**

Otto-von-Guericke Universitaet Magdeburg
Faculty of Mathematics
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Max Planck Institute for Dynamics of Complex Technical Systems
Computational Methods in Systems and Control Theory
Magdeburg, Germany

benner@mpi-magdeburg.mpg.de feng@mpi-magdeburg.mpg.de

http://www.mpi-magdeburg.mpg.de/csc/teaching/17ss/mor

Outline

Image Compression by Truncated SVD

- A digital image with $n_x \times n_y$ pixels can be represented as matrix $X \in \mathbb{R}^{n_x \times n_y}$, where x_{ij} contains color information of pixel (i, j).
- Memory (in single precision): $4 \cdot n_x \cdot n_y$ bytes.

Theorem (Schmidt-Mirsky/Eckart-Young)

Best rank-r approximation to $X \in \mathbb{R}^{n_x \times n_y}$ w.r.t. spectral norm:

$$X \approx \widehat{X} = \sum_{j=1}^{r} \sigma_j u_j v_j^T,$$

where $X = U \Sigma V^T$ is the singular value decomposition (SVD) of X.
The approximation error is $\left\| X - \widehat{X} \right\|_2 = \sigma_{r+1}$.

Idea for dimension reduction

Instead of X save $u_1, \dots, u_r, \sigma_1 v_1, \dots, \sigma_r v_r$.
$\rightsquigarrow$ memory $= 4r \times (n_x + n_y)$ bytes.

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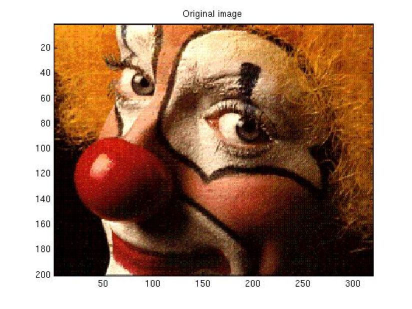
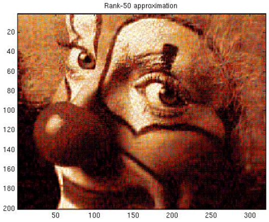
Example: Clown



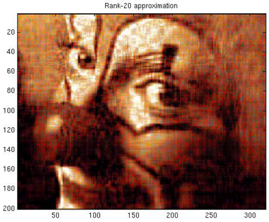
320×200 pixel

$\rightsquigarrow \quad \approx 256$ kB

- rank $r = 50$, ≈ 104 kB



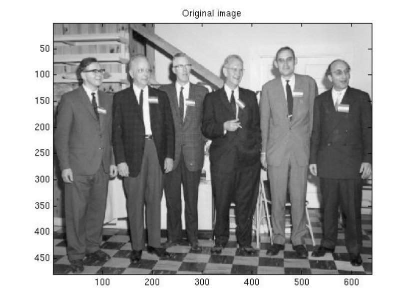
- rank $r = 20$, ≈ 42 kB



Example: Gatlinburg

Organizing committee
Gatlinburg/Householder Meeting 1964:

James H. Wilkinson, Wallace Givens, George Forsythe, Alston Householder, Peter Henrici, Fritz L. Bauer.



640×480 pixel, ≈ 1229 kB

Dimension Reduction via SVD

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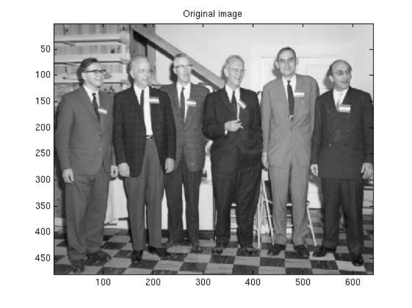
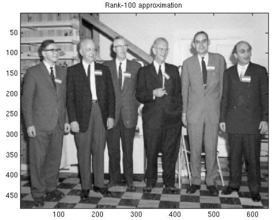


640 × 480 pixel, ≈ 1229 kB

- rank $r = 100$, ≈ 448 kB



- rank $r = 50$, ≈ 224 kB

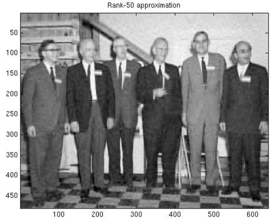
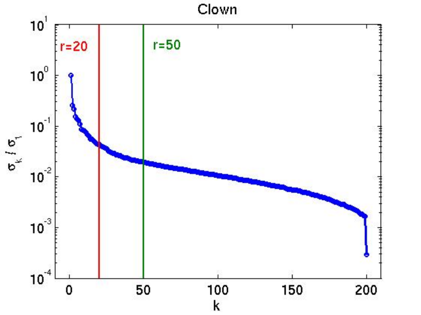
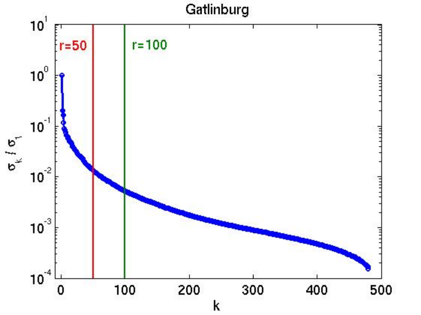


Image data compression via SVD works, if the singular values decay (exponentially).

Singular Values of the Image Data Matrices





A different viewpoint

Linear Mapping

A matrix $A \in \mathbb{R}^{l \times k}$ represents a linear mapping

$$\mathcal{A} : \mathbb{R}^k \mapsto R^l : x \mapsto := Ax.$$

The truncated SVD ignores small Hankel singular values and thus the related left and right singular vectors.
Consequence:

- Vectors (almost) in the kernel of A do not contribute to $\mathrm{range}(A)$ and can hardly or not at all be reconstructed from the input-output relation ("A^{-1}") $\rightarrow$ "unobservable" states.
- Vectors (almost) in $(\mathrm{range}(A))^{\perp}$ cannot be "reached" from any $x \in \mathbb{R}^k$ $\rightarrow$ "unreachable/uncontrollable" states.
- Hence, the truncated SVD ignores states hard to reconstruct and hard to reach.

The Laplace transform

Definition

The Laplace transform of a time domain function $f \in L_{1,\mathrm{loc}}$ with $\mathrm{dom}\,(f) = \mathbb{R}_0^+$ is

$$\mathcal{L} : f(t) \mapsto F(s) := \mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt, \quad s \in \mathbb{C}.$$

F is a function in the (Laplace or) frequency domain.

Note: for frequency domain evaluations ("frequency response analysis"), one takes $\mathrm{re}\, s = 0$ and $\mathrm{im}\, s \geq 0$. Then $\omega := \mathrm{im}\, s$ takes the role of a frequency (in [rad/s], i.e., $\omega = 2\pi v$ with v measured in [Hz]).

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The Laplace transform

Lemma

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s) - f(0).$$

if f(0)=0, then

$$\mathcal{L}\{\dot{f}(t)\}(s) = sF(s).$$

Note: For ease of notation, in the following we will use lower-case letters for both, a function $f(t)$ and its Laplace transform $F(s)$!

The Model Reduction Problem as Approximation Problem in Frequency Domain

Linear Systems in Frequency Domain

Application of Laplace transform $(x(t) \mapsto x(s),\ \dot{x}(t) \mapsto sx(s))$ to linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

with $x(0) = 0$ yields:

$$sEx(s) = Ax(s) + Bu(s), \quad y(s) = Cx(s) + Du(s),$$

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$\Longrightarrow$ I/O-relation in frequency domain:

$$y(s) = \Big(\underbrace{C(sE - A)^{-1}B + D}_{=:G(s)} \Big) u(s).$$

$G(s)$ is the **transfer function** of Σ.

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Goal: Fast evaluation of mapping $u \rightarrow y$.

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The Model Reduction Problem as Approximation Problem in Frequency Domain

Formulating model reduction in time domain

Approximate the dynamical system

$$\begin{array}{rcl} E\dot{x} & = & Ax + Bu, \qquad E, A \in \mathbb{R}^{n\times n},\; B \in \mathbb{R}^{n\times m}, \\ y & = & Cx + Du, \qquad C \in \mathbb{R}^{q\times n},\; D \in \mathbb{R}^{q\times m}, \end{array}$$

by reduced-order system

$$\begin{array}{rcl} \hat{E}\dot{\hat{x}} & = & \hat{A}\hat{x} + \hat{B}u, \quad \hat{E}, \hat{A} \in \mathbb{R}^{r\times r},\; \hat{B} \in \mathbb{R}^{r\times m}, \\ \hat{y} & = & \hat{C}\hat{x} + \hat{D}u, \quad \hat{C} \in \mathbb{R}^{q\times r},\; \hat{D} \in \mathbb{R}^{q\times m} \end{array}$$

of **order** $r \ll n$**,** such that

$$||y - \hat{y}|| = \left|\left|Gu - \hat{G}u\right|\right| \leq \left|\left|G - \hat{G}\right|\right| \cdot ||u|| < \text{tolerance} \cdot ||u|| \,.$$

Properties of linear systems

Definition

A linear system

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)$$

is **stable** if its transfer function $G(s)$ has all its poles in the left half plane and it is **asymptotically (or Lyapunov or exponentially)** stable if all poles are in the open left half plane $\mathbb{C}^- := \{z \in \mathbb{C} \,|\, \Re(z) < 0\}$.

Lemma

Sufficient for asymptotic stability is that A is **asymptotically stable** (or **Hurwitz**), i.e., the eigenvalues of the generalized eigenvalue problem $Ax = \lambda Ex$, denoted by $\Lambda(A, E)$, satisfies $\Lambda(A, E) \subset \mathbb{C}^-$.

Note that by abuse of notation, often *stable system* is used for asymptotically stable systems.

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Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

For a linear (time-invariant) system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t), \end{cases} \qquad \text{with transfer function} \quad G(s) = C(sI - A)^{-1}B + D,$$

the quadruple $(A, B, C, D) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called a **realization** of Σ.

Realizations are not unique!

Transfer function is invariant under **state-space transformations**,

$$\mathcal{T} : \begin{cases} x \to Tx, \\ (A, B, C, D) \to (TAT^{-1}, TB, CT^{-1}, D), \end{cases}$$

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Realizations are not unique!

Transfer function is invariant under addition of uncontrollable/unobservable states:

$$\frac{d}{dt}\begin{bmatrix} x \\ x_1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + \begin{bmatrix} B \\ B_1 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & 0 \end{bmatrix}\begin{bmatrix} x \\ x_1 \end{bmatrix} + Du(t),$$

$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u(t), \quad y(t) = \begin{bmatrix} C & C_2 \end{bmatrix}\begin{bmatrix} x \\ x_2 \end{bmatrix} + Du(t),$$

for arbitrary $A_j \in \mathbb{R}^{n_j \times n_j}$, $j = 1, 2$, $B_1 \in \mathbb{R}^{n_1 \times m}$, $C_2 \in \mathbb{R}^{q \times n_2}$ and any $n_1, n_2 \in \mathbb{N}$.

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Realizations are not unique!

Hence,

$$(A, B, C, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_1 \end{bmatrix}, \begin{bmatrix} B \\ B_1 \end{bmatrix}, \begin{bmatrix} C & 0 \end{bmatrix}, D \right),$$

$$(TAT^{-1}, TB, CT^{-1}, D), \qquad \left(\begin{bmatrix} A & 0 \\ 0 & A_2 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix}, \begin{bmatrix} C & C_2 \end{bmatrix}, D \right),$$

are all realizations of Σ!

Realizations of Linear Systems (with $E = I_n$ for simplicity)

Definition

The **McMillan degree** of Σ is the unique minimal number $\hat{n} \geq 0$ of states necessary to describe the input-output behavior completely.
A **minimal realization** is a realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of Σ with order $\hat{n}$.

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Theorem

A realization (A, B, C, D) of a linear system is minimal $\iff$
(A, B) is controllable and (A, C) is observable.